Chromatic Seminar
Galois Descent, Picard Groups and the $K(n)$-local Category

Shay Ben Moshe

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A great source for all things in chromatic homotopy, especially for getting references, is [BB19]. The following thesis is very useful for this talk [Hea14].

1 Galois Descent

Let $A \to B$ be a Galois field extension, with Galois group $G = \mathrm{Gal}(B/A)$. By definition we know that $B^G = A$.

Given a module $M \in \mathrm{Mod}_A$, we can extend scalars and form $N = B \otimes_A M \in \mathrm{Mod}_B$, by considering B as a (B, A)-bi-module. Moreover, we can define an action of G on $B \otimes_A M$ by its action on B, i.e. $g(b \otimes m) = gb \otimes m$. However, this action is not B-linear, for $b' \in B$ we have $g(b'(b \otimes m)) = g(b'b \otimes m) = g(b'b) \otimes m = g(b')g(b) \otimes m = g(b')(g(b) \otimes m)$, i.e. $g(b'n) = g(b')g(n)$, namely, it is *semilinear*. So $B \otimes_A M$ is a B-module with a semilinear G-action, we denote this category by Mod_B^G. Thus we got a functor $B \otimes_A - : \mathrm{Mod}_A \to \mathrm{Mod}_B^G$.

Given $N \in \mathrm{Mod}_B^G$, we can take the fixed points $M = N^G$. This has the structure of an A-module, since for $m \in M$ and $a \in A$, and any $g \in G$, we have $g(am) = g(a)g(m) = am$, i.e. am is a fixed points as well. This gives a functor $(-)^G : \mathrm{Mod}_B^G \to \mathrm{Mod}_A$.

Theorem 1 (Hilbert 90)**.** *The two functors form an equivalence of categories* $B \otimes_A - : \mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_B^G : (-)^G$.

Before saying anything about the proof, I would like to generalize a little bit, which will both be useful later and put it in a more suitable context.

2 Galois Extensions of Commutative Rings

We would like to generalize the notion of a Galois extension of fields to commutative rings (and later E_∞-rings in arbitrary categories), so we first make some remarks on the situation of Galois extensions of fields. We started with a map $A \to B$, making B into an A-algebra, and $G = \mathrm{Gal}\,(B/A)$ acts via A-algebra maps. Then, we get a morphism to the fixed points, which is an isomorphism $A \xrightarrow{\sim} B^G$.

Consider the morphism $B \otimes_A B \to \prod_G B$ given by $x \otimes y \mapsto (xg\,(y))_g$.

Proposition 2. *Under the hypothesis above, $A \to B$ Galois is equivalent to $B \otimes_A B \xrightarrow{\sim} \prod_G B$.*

Proof. Assume that $A \to B$ is Galois. Then by the primitive element theorem, $B = A\,(\alpha)$. Consider its minimal polynomial f, whose roots are $g\,(\alpha)$, i.e. in B we have $f\,(x) = \prod_G (x - g\,(\alpha))$. We get:

$$\begin{aligned} B \otimes_A B &= B \otimes_A A\,[x]\,/\,f \\ &= B\,[x]\,/\,f \\ &= B\,[x]\,/\left(\prod_G (x - g\,(\alpha))\right) \\ &= \prod_G B\,[x]\,/\,(x - g\,(\alpha)) \\ &= \prod_G B \end{aligned}$$

For the other direction take $\dim_A$ to both sides to get $(\dim_A B)^2 = |G| \dim_A B$, so $|G| = \dim_A B$, which implies that the extension is Galois. $\square$

This is very generalizable, as follows:

Definition 3 ([Rog05])**.** Let $A \to B$ be a map of commutative rings. Let G act on B via A-algebra morphisms. We say that $A \to B$ is a *G-Galois extension* if:

1. $A \xrightarrow{\sim} B^G$,
2. $B \otimes_A B \xrightarrow{\sim} \prod_G B$.

2.1 Galois Descent, Again

In this situation we can repeat the constructions from before and to get

Theorem 4 (Hilbert 90)**.** $B \otimes_A - : \mathrm{Mod}_A \rightleftarrows \mathrm{Mod}_B^G : (-)^G$ *is an equivalence.*

A way to prove this is as follows. Generally, given a morphism of rings $f : A \to B$ one can form a category of modules over B together with *descent data* Desc_f, and a functor $\mathrm{Mod}_A \to \mathrm{Desc}_f$. In many situations this is an equivalence, for example when f is faithfully flat, as it is in field extensions. Furthermore, if $A \to B$ is a G-Galois extension, then Desc_f is equivalent to Mod_B^G.

There are various reasons for this being useful. For example, it might turn out that modules over B are easier to handle. Furthermore, we can also forget the G-action, and obtain information on $\mathrm{Mod}_A \cong \mathrm{Mod}_B^G$ from Mod_B via the forgetful functor or its adjoint.

3 Galois Extensions of E_∞-Rings

In fact, we can mimic this whole story in the homotopical world. Let $\mathcal{C}$ be some symmetric monoidal ∞-category (e.g. Sp or Sp_E).

Definition 5. Let $A \to B$ be a map of E_∞-rings in $\mathcal{C}$. Let G act on B via A-algebra morphisms. We say that $A \to B$ is a *G-Galois extension* if:

1. $A \xrightarrow{\sim} B^{hG}$,
2. $B \otimes_A B \xrightarrow{\sim} \prod_G B$.

3.1 Galois Descent, Once More

To define the corresponding ∞-category $\mathrm{Mod}_B^G(\mathcal{C})$, we reinterpret Mod_B^G from before. First, we note that we have a G action on the category Mod_B: given $N \in \mathrm{Mod}_B$ and $g \in G$ we define $N^g \in \mathrm{Mod}_B$ to have the same elements and action by $B \otimes N \overset{g \otimes \mathrm{id}}{\to} B \otimes N \to N$ i.e. $b._g n = g(b)\, n$. Consider the fixed-point category (in the 2-category of categories) Mod_B^G, an object here is $N \in \mathrm{Mod}_B$ together with for every $g \in G$ an isomorphism $\phi_g : N \to N^g$. Being B-linear means $\phi_g(bn) = b._g \phi_g(n) = g(b)\, \phi_g(n)$, which is precisely a semilinear action. Thus Mod_B^G is the fixed-point category of the action of G on Mod_B. In much the same way, we have a G-action on $\mathrm{Mod}_B(\mathcal{C})$, and we define $\mathrm{Mod}_B^G(\mathcal{C}) = (\mathrm{Mod}_B(\mathcal{C}))^{hG}$, the ∞-category of modules over B in $\mathcal{C}$ together with a semilinear action of G.

Then, assuming $A \to B$ is a faithful (which is true for ordinary rings, but not known for general E_∞-rings), we have:

Theorem 6 (Hilbert 90 [Ban17, Theorem 2.8])**.** *$B \otimes_A - : \mathrm{Mod}_A(\mathcal{C}) \rightleftarrows \mathrm{Mod}_B^G(\mathcal{C}) : (-)^G$ is an equivalence.*

4 Galois Extensions in Chromatic Homotopy

We first recall the construction of E_n, Morava E-theory at height n. Let k be a perfect field of characteristic p, and Γ a formal group law over k of height n. Then, Lubin-Tate developed a deformation theory for Γ/k, and proved that there is a *universal deformation*, i.e. a ring $\mathrm{LT}_{k,\Gamma}$ and Γ_U over it. Moreover, non-canonically $\mathrm{LT}_{k,\Gamma} = Wk\,[[u_1, \ldots, u_{n-1}]]$. We then add a variable u at degree 2 $\mathrm{LT}_{k,\Gamma}\left[u^{\pm 1}\right]$, and twist by it to get a formal group law of the correct degree $u\Gamma_U$. One can then easily finds that this is Landweber exact, so using LEFT we define $E\left(k,\Gamma\right)_*(X) = \mathrm{BP}_*(X) \otimes_{\mathrm{BP}_*} \mathrm{LT}_{k,\Gamma}\left[u^{\pm 1}\right]$. This defines a spectrum $E\left(k,\Gamma\right)$ which we call Morava E-theory at height n.

This depends on k, Γ, and there are many possible choices for them. Conceptually, it is nicest to choose $k = \overline{\mathbb{F}}_p$ (as then E_n is the maximal unramified extension), but for computations it is easier to choose $k = \mathbb{F}_{p^n}$, over which all automorphisms are define, so we work with it. We write E_n or even E for $E\left(\mathbb{F}_{p^n}, \Gamma\right)$.

By Goerss-Hopkins-Miller ([GH04; GH05]), the spectrum $E\left(k,\Gamma\right)$ can be upgraded to an E_∞-ring, functorially in k, Γ. The functoriality means that $\mathbb{G}_n = \mathrm{Aut}\left(\mathbb{F}_{p^n}, \Gamma\right)$ (i.e. an isomorphism $\varphi : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$, and a strict isomorphism $\Gamma \to \varphi^*\Gamma$) acts on E_n through E_∞-ring maps, and in fact they show that $\mathbb{G}_n \xrightarrow{\sim} \mathrm{Aut}_{E_\infty}\left(E_n\right)$ (it should be noted that this group is infinite and acts continuously in some sense, but we will ignore this point completely). This automorphism group is called the *Morava Stabilizer Group.* We note that we have the homomorphisms that fix the field, that is $S_n = \mathrm{Aut}\left(\Gamma\right)$, and that we have Galois part, in fact $\mathbb{G}_n = S_n \rtimes \mathrm{Gal}\left(\mathbb{F}_{p^n}/\mathbb{F}_p\right)$.

By a standard argument, E_n is $K\left(n\right)$-local (this shouldn't be too surprising, as on the side of the algebra, $\mathrm{LT}_{k,\Gamma}$ (roughly E_n) is the deformation of the point $\mathbb{F}_{p^n}, \Gamma$ (roughly $K\left(n\right)$) in $\mathcal{M}_{\mathrm{fg}}$). By adjunction, this means that we have a map $\mathbb{S}_{K(n)} \to E_n$ (where $\mathbb{S}_{K(n)} = L_{K(n)}\mathbb{S}$ is the $K\left(n\right)$-local sphere) and the $\mathbb{G}_n$ action is over $\mathbb{S}_{K(n)}$, and it makes sense to work in the category $\mathcal{C} = \mathrm{Sp}_{K(n)}$ of $K\left(n\right)$-local spectra from now on. Another big theorem by Devinatz-Hopkins ([DH04], see also [Rog05, section 5.4]), is that indeed this is an $\mathbb{G}_n$-Galois extension, i.e. $\mathbb{S}_{K(n)} = E_n^{h\mathbb{G}_n}$.

Before moving on, let's recap and write a table of analogies:

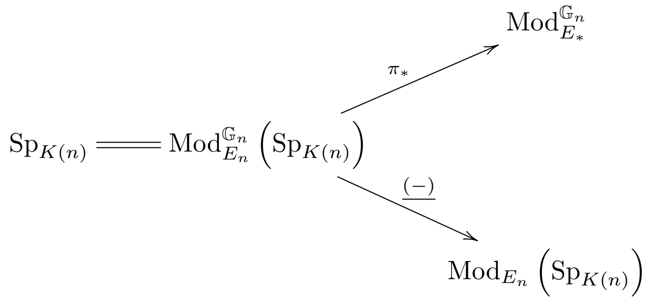
	Algebra	**Homotopy**
Unit	$\mathbb{Z}$	$\mathbb{S}$
Point in Spec of the unit	$\mathbb{F}_p$	$K\left(n\right)$
Localized unit	$\mathbb{Z}_p = W\mathbb{F}_p$	$\mathbb{S}_{K(n)}$
Extension	$\mathbb{Z}_p\left(\zeta_{p^n-1}\right) = W\mathbb{F}_{p^n}$	E_n
Galois group	$G = \mathrm{Gal}\left(\mathbb{F}_{p^n}/\mathbb{F}_p\right)$	$\mathbb{G}_n$
	$\left(W\mathbb{F}_{p^n}\right)^G = W\mathbb{F}_p$	$E_n^{\mathbb{G}_n} = \mathbb{S}_{K(n)}$

4.1 Galois Descent, Last Time - Ninja Turtles, Morava Modules etc.

We now wish to describe the situation of the relevant module categories. Recall that we work with $\mathcal{C} = \mathrm{Sp}_{K(n)}$, $A = \mathbb{S}_{K(n)}$ (thus $\mathrm{Mod}_A(\mathcal{C}) = \mathrm{Sp}_{K(n)}$), $B = E_n$ and $G = \mathbb{G}_n$. By [Rog05, Prop. 5.4.9] the extension $E_n/\mathbb{S}_{K(n)}$ is faithful, thus we are under the conditions of Hilbert 90 so we get

Theorem 7 (For another approach see [Mat16, Proposition 10.10])**.** *There is an equivalence* $E_n \otimes_{K(n)} - : \mathrm{Sp}_{K(n)} \rightleftarrows \mathrm{Mod}_{E_n}^{\mathbb{G}_n}\left(\mathrm{Sp}_{K(n)}\right) : (-)^{\mathbb{G}_n}$ *between the* $K(n)$*-local category to the category of* spectral Morava modules *(i.e.* $K(n)$*-local* E_n*-modules with a* $\mathbb{G}_n$*-semilinear action).*

Definition 8. We also take the homotopy groups, to the category of *Morava modules* $\mathrm{Mod}_{E_*}^{\mathbb{G}_n}$ (complete $(E_n)_*$-modules with a $\mathbb{G}_n$-semilinear action.) In another direction, we can forget the $\mathbb{G}_n$ action to the category of *Ninja turtles* $\mathrm{Mod}_{E_n}\left(\mathrm{Sp}_{K(n)}\right)$ ($K(n)$-local E_n-modules).



These constructions have many uses, we will demonstrate one of them via Picard groups.

Remark 9. Note that our tensor product is in the $K(n)$-local category, i.e. $L_{K(n)}(E_n \otimes X)$. For example, the map to Morava modules gives $E_*^\vee(X) = \pi_*\left(L_{K(n)}(E_n \otimes X)\right)$, this is not a homology theory, but as we can see it plays very nicely in this framework.

5 Picard Groups

Definition 10. Let $\mathcal{C}$ be a symmetric monoidal (∞-)category. An object $M \in \mathcal{C}$ is called *invertible* if there exists an object M^{-1} and an equivalence $M \otimes M^{-1} \cong 1_{\mathcal{C}}$. The *Picard group* $\mathrm{Pic}(\mathcal{C})$ is the collection of invertible objects up to isomorphism, with tensor product as the group structure (abelian, since we assumed symmetric monoidal).

We will not need this in the rest of the talk, but instead of taking the quotient by isomorphisms, we can look at the space (i.e. (∞-)groupoid) of invertible objects $\mathbf{Pic}(\mathcal{C})$. This has an E_∞ structure of by the (symmetric) tensor product, which is group-like by assumption. A group-like E_∞-space is also known as a (connective) spectrum, so we can consider $\mathbf{Pic}(\mathcal{C})$ as a spectrum. By definition, $\pi_0 \mathbf{Pic}(\mathcal{C}) = \mathrm{Pic}(\mathcal{C})$.

Remark 11. If M is invertible, then it is in particular dualizable, and if the category is also closed, then we get that $M^{-1} = DM = \hom(M, 1_{\mathcal{C}})$.

Classically this was considered in the context of algebraic geometry, where for a scheme X we look at the category of quasi-coherent sheaves over X, i.e. $\mathrm{Pic}(X) = \mathrm{Pic}(\mathrm{QCoh}(X))$. In the affine case we get $\mathrm{Pic}(R) = \mathrm{Pic}(\mathrm{Mod}_R)$. This captures interesting information known in many names (e.g. the first cohomology, ideal class group...)

These constructions are interesting for various reasons. For example, this is a way to construct (possibly interesting) groups and spectra, which are invariants of $\mathcal{C}$. Furthermore, clearly $\mathbb{S}^n \in \mathrm{Pic}(\mathbb{S}) = \mathrm{Pic}(\mathrm{Sp})$, and we are used to the fact that $\mathbb{S}^n \otimes -$ (i.e. de/suspension) is an important operation on spectra. Similarly, for any $M \in \mathrm{Pic}(\mathcal{C})$ we get an invertible operation on $\mathcal{C}$, namely $M \otimes -$.

Example 12. For a usual ring R, a module M is invertible if and only if it is finite locally free module of rank 1. For example, for $R = \mathbb{Z}$ there is only $\mathbb{Z}$ itself, thus $\mathrm{Pic}(\mathbb{Z}) = \{\mathbb{Z}\}$.

Example 13. In any stable ∞-category $\mathcal{C}$ we have $\Sigma^n 1_{\mathcal{C}} \in \mathrm{Pic}(\mathcal{C})$, so we have a map $\mathbb{Z} \to \mathrm{Pic}(\mathcal{C})$. We claim that in $\mathrm{Pic}(\mathrm{Sp})$ this is an isomorphism.

Proof. Assume that $M \in \mathrm{Pic}(\mathbb{S})$. Since $M \otimes M^{-1} = \mathbb{S}$, for $k = \mathbb{F}_p$ or $k = \mathbb{Q}$ we get by Kunneth that $H_*(M; k) \otimes_k H_*\left(M^{-1}; k\right) = H_*(\mathbb{S}; k) = k$, thus $H_*(M; k)$ is in a single degree. Using the universal coefficient theorem we deduce that $H_*(M; \mathbb{Z})$ is in a single degree, say m. By (stable) Hurewicz we get $\pi_m(M) = \mathbb{Z}$, so have a map $\mathbb{S}^m \to M$ inducing an isomorphism on homology. Since the spheres are H$\mathbb{Z}$-local, it will suffice to show that M is H$\mathbb{Z}$-local. Let X be H$\mathbb{Z}$-acyclic, then $X \otimes M^{-1}$ is also H$\mathbb{Z}$-acyclic, and we get $\hom(X, M) = \hom\left(X \otimes M^{-1}, \mathbb{S}\right) = 0$ (as $\mathbb{S}$ is H$\mathbb{Z}$-local). ☐

6 $K(n)$-local Picard Group

As we have seen, $\mathrm{Pic}(\mathbb{S}) = \mathbb{Z}$, but one may wonder about $\mathrm{Pic}_n = \mathrm{Pic}\left(\mathrm{Sp}_{K(n)}\right)$. It turns out that this group contains many interesting elements.

Using the maps $\mathrm{Sp}_{K(n)} \xrightarrow{\sim} \mathrm{Mod}_{E_n}^{\mathbb{G}_n}\left(\mathrm{Sp}_{K(n)}\right) \to \mathrm{Mod}_{E_*}^{\mathbb{G}_n}$ we get a map on Picard groups, $\varepsilon_n : \mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}} = \mathrm{Pic}\left(\mathrm{Mod}_{E_*}^{\mathbb{G}_n}\right)$ (which is many times not injective, and it is still open if it is always surjective). We also denote the kernel by $\kappa_n = \ker \varepsilon_n$. Before moving on, we table some known cases of these, to see some of the phenomena going on:

n	p	Pic_n	$\mathrm{Pic}_n^{\mathrm{alg}}$	κ_n	
1	≥ 3	$\mathbb{Z}_p \times \mathbb{Z}/2\,(p-1)$	$\mathbb{Z}_p \times \mathbb{Z}/2\,(p-1)$	0	
1	2	$\mathbb{Z}_2 \times \mathbb{Z}/2 \times \mathbb{Z}/4$	$\mathbb{Z}_2 \times (\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	
2	≥ 5	$\mathbb{Z}_p^2 \times \mathbb{Z}/2\left(p^2-1\right)$	$\mathbb{Z}_p^2 \times \mathbb{Z}/2\left(p^2-1\right)$	0	
2	3	$\mathbb{Z}_3^2 \times \mathbb{Z}/2\left(3^2-1\right) \times (\mathbb{Z}/3)^2$	$\mathbb{Z}_3^2 \times \mathbb{Z}/2\left(3^2-1\right)$	$(\mathbb{Z}/3)^2$	
2	2	?	?	?	(in progress)

As we can see, there are many interesting phenomena here. We shall state some of the main known results, and then move on to explicitly computing the case $n = 1$ and $p \geq n$.

Theorem 14 ([HMS92, section 9])**.** *The map $\mathbb{Z} \to \mathrm{Pic}_n$ given by $m \mapsto \mathbb{S}_{K(n)}^m$ can be extended to an injective map $\mathbb{Z}_p \to \mathrm{Pic}_n$.*

Theorem 15 ("Large primes" [HMS92, Proposition 7.5])**.** *If $2\,(p-1) \geq n^2$ and $p-1 \nmid n$ (i.e. $p > 2$), then $\varepsilon_n : \mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}}$ is injective.*

Proof sketch. In this range the $K(n)$-local E_n-Adams spectral sequence collapses for degree reasons at the E_2-page for any $X \in \mathrm{Pic}_n$. The isomorphism $E_*^\vee X \cong E_*^\vee \mathbb{S}_{K(n)}$ induces an isomorphism on the E_2-page, which gives us a map $\mathbb{S}_{K(n)} \to X$ that induces the isomorphism on the E_2-page, i.e. gives an equivalence of spectra. □

Theorem 16 ("Larger primes" [Pst18, Theorem 1.1])**.** *If $2\,(p-1) \geq n^2 + n$ then $\varepsilon_n : \mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}}$ is an isomorphism.*

Since E_* is 2-periodic, and $\mathbb{G}_n$ acts level-wise, we can look at subgroup of index 2, of modules which are concentrated in even degrees. This gives a short exact sequence $0 \to \mathrm{Pic}_n^0 \to \mathrm{Pic}_n \to \mathbb{Z}/2 \to 0$, and similarly for the algebraic Picard. The map restricts to $\varepsilon_n : \mathrm{Pic}_n^0 \to \mathrm{Pic}_n^{\mathrm{alg},0}$.

Very generally, let G be a group acting on a ring R. There is a forgetful functor $\mathrm{Mod}_R^G \to \mathrm{Mod}_R$, giving $\mathrm{Pic}\left(\mathrm{Mod}_R^G\right) \to \mathrm{Pic}\left(\mathrm{Mod}_R\right)$, we denote its kernel by K_R^G. Then, we have a map $K_R^G \to H^1\left(G; R^\times\right)$: given $M \in K_R^G$, because it is in the kernel it has a generator e, and for any $g \in G$, ge is another generator, so they differ by $u_g \in R^\times$, which defines a cohomology class $u : G \to R^\times$. In fact, this map is an isomorphism (this is classical, but can be found in [HMS92, Proposition 8.4], this also connects what we call Hilbert 90 with usual cohomological statement). See Tomer's remarks below for a more general perspective. In our case, we deduce that:

Theorem 17. $\mathrm{Pic}_n^{\mathrm{alg},0} \cong H^1\left(\mathbb{G}_n; W\mathbb{F}_{p^n}\left[\left[u_1, \ldots, u_{n-1}\right]\right]^\times\right)$.

6.1 Picard at Height 1 and Primes ≥ 3

We don't really have time to do this, but let's sketch the argument without using the theorems above (details are in [HMS92, Section 2].)

Let's recall some facts about height 1. This is one of the places where it is easier to use $k = \mathbb{F}_{p^n} = \mathbb{F}_p$ rather than $\overline{\mathbb{F}}_p$. In this case $E = E_1 = \mathrm{KU}_p$ is p-complete complex K-theory, whose homotopy groups are $\mathbb{Z}_p\left[u^{\pm 1}\right]$. Further $\mathbb{G}_1 = S_1 = \mathbb{Z}_p^\times$ are the Adams operations ψ^k. Since $\mathbb{Z}_p^\times$ is also topologically cyclic it has a generator g, corresponding to ψ^g. Then we get $\mathbb{S}_{K(1)} = E^{h\mathbb{Z}_p^\times} = \mathrm{Fib}\left(E \xrightarrow{\psi^g - 1} E\right)$.

Recall that we have a short exact sequence $0 \to \mathrm{Pic}_n^0 \to \mathrm{Pic}_n \to \mathbb{Z}/2 \to 0$. Modifying the construction above, for every invertible $\lambda \in \mathbb{Z}_p$ we can define $X_\lambda = \mathrm{Fib}\left(E \xrightarrow{\psi^g - \lambda} E\right)$, and for $\lambda = g^n$ one gets $X_{g^n} = \mathbb{S}_{K(1)}^{2n}$. We don't have the time to prove the following result, but it isn't too hard.

Proposition 18. *The map $\mathbb{Z}_p^\times \to \mathrm{Pic}_n^0$ given by $\lambda \mapsto X_\lambda$ is an isomorphism.*

Proposition 19. *The sequence $0 \to \mathrm{Pic}_n^0 \to \mathrm{Pic}_n \to \mathbb{Z}/2 \to 0$ doesn't split.*

Proof. Assume by negation that that it does split. $\mathbb{S}_{K(1)}^{-1}$ represents $1 \in \mathbb{Z}/2$, so $X_\mu \otimes \mathbb{S}_{K(1)}^{-1}$ is of order 2 for some μ. Then we get that $\mathbb{S}_{K(1)} = \left(X_\mu \otimes \mathbb{S}_{K(1)}^{-1}\right)^{\otimes 2} = X_{\mu^2} \otimes \mathbb{S}_{K(1)}^{-2}$, i.e. $X_{\mu^2} = \mathbb{S}_{K(1)}^2 = X_g$. Thus, by the isomorphism from before $g = \mu^2$, which contradicts the fact that g is a generator. □

Corollary 20. $\mathrm{Pic}_n \cong \mathbb{Z}_p \times \mathbb{Z}/2\,(p-1)$.

Proof. Since $\mathrm{Pic}_n^0 \cong \mathbb{Z}_p \times \mathbb{Z}/\,(p-1)$, and the sequence doesn't split, we get that the only extension is the desired extension. □

6.2 Determinant Sphere

There are some special elements in the Picard, and we give an example of one of them. Recall that $\mathbb{G}_n = S_n \rtimes \mathrm{Gal}\left(\mathbb{F}_{p^n}/\mathbb{F}_p\right)$ where $S_n = \mathrm{Aut}\,(\Gamma)$. Consider $\mathcal{O} = \mathrm{End}\,(\Gamma)$, then $S_n = \mathcal{O}^\times$ acts on $\mathcal{O}$ by composition. These things can be described very concretely, and then one sees that $\mathcal{O}$ is a (free) $W\mathbb{F}_{p^n}$-module of rank n. This means that we get a map $S_n \xrightarrow{\text{action}} \mathrm{GL}_n\left(W\mathbb{F}_{p^n}\right) \xrightarrow{\det} \left(W\mathbb{F}_{p^n}\right)^\times$ (in fact it is Galois invariant and thus lands in $\mathbb{Z}_p^\times$). We define the composition $\det : \mathbb{G}_n \to S_n \to \left(W\mathbb{F}_{p^n}\right)^\times \subseteq \left(E_n\right)_0^\times$. This map is a cohomology class in $H^1\left(\mathbb{G}_n; W\mathbb{F}_{p^n}\left[\left[u_1, \ldots, u_{n-1}\right]\right]^\times\right)$. As the latter is isomorphic to $\mathrm{Pic}_n^{\mathrm{alg},0}$, the map det corresponds to a Morava module $E_*\langle\det\rangle \in \mathrm{Pic}_n^{\mathrm{alg}}$. In [BBGS18] they show that it can be lifted to an element $\mathbb{S}\langle\det\rangle \in \mathrm{Pic}_n$ itself (for all n and p), i.e. $E_*^\vee \mathbb{S}\langle\det\rangle = E_*\langle\det\rangle$.

This element $\mathbb{S}\langle\det\rangle$ is also very related to I_n of Gross-Hopkins duality (namely, for p large enough $I_n = \Sigma^{n^2-n}\mathbb{S}\langle\det\rangle$, and for smaller primes you need to tensor with another spectrum), but as we haven't discussed this duality and don't have more time, we will not pursue this train of thought.

7 Tomer's Remarks

Tomer made a few remarks. I don't remember all of them, but here are some.

Here is the way Picard groups are really computed. Recall that in fact we have a Picard spectrum $\mathbf{Pic}(\mathcal{C})$, and $\mathrm{Pic}(\mathcal{C}) = \pi_0 \mathbf{Pic}(\mathcal{C})$. Assume that $\mathcal{C}$ has a G-action (i.e. we have a BG diagram of ∞-categories), then $\mathbf{Pic}(\mathcal{C})^{hG} = \mathbf{Pic}\left(\mathcal{C}^{hG}\right)$ (in fact, $\mathbf{Pic}$ is the right adjoint to the inclusion of ∞-groups in monoidal categories, thus it commutes with limits). This gives a fixed-point spectral sequence $E_2^{n,s} = H^s(G; \pi_{n+s}(\mathbf{Pic}(\mathcal{C}))) \Rightarrow \pi_n\left(\mathbf{Pic}\left(\mathcal{C}^{hG}\right)\right)$.

Related to this, note that $\pi_1(\mathbf{Pic}(\mathrm{Mod}_R)) = R^\times$, and since this is a 1-category $\mathbf{Pic}$ doesn't have higher homotopy groups, so in the spectral sequence above, the elements in $n = 0$ (i.e. giving contribution to $\mathrm{Pic}\left(\mathrm{Mod}_R^G\right)$) are $H^0(G; \pi_0(\mathbf{Pic}(\mathrm{Mod}_R))) = (\mathrm{Pic}(\mathrm{Mod}_R))^G$ and $H^1(G; \pi_1(\mathbf{Pic}(\mathrm{Mod}_R))) = H^1(G; R^\times)$, which explains the relationship between first cohomology and the kernel K_R^G.

Another interesting thing to note is that we have seen that Pic_n has a copy of $\mathbb{Z}/2(p^n-1)$ for the cases we have tabled. This can be seen as follows. We can take $K(n)$-homology $\mathrm{Sp}_{K(n)} \to \mathrm{Mod}_{K(n)_*}$. Since $K(n)_*$ is a $2(p^n-1)$-periodic graded field, the only invertible modules (up to isomorphism) are $\Sigma^k K(n)$ for $k = 0, 1, \ldots, 2(p^n-1)-1$, thus $\mathrm{Pic}\left(\mathrm{Mod}_{K(n)_*}\right) = \mathbb{Z}/2(p^n-1)$, and this is the copy we have seen.

Lastly, $\mathrm{Pic}(\mathcal{C})$ comes equipped with a natural topology. Given an element $M \in \mathrm{Pic}(\mathcal{C})$, we get an automorphism $M \otimes - : \mathcal{C} \to \mathcal{C}$, which means that it sends compact to objects to themselves $M \otimes - : \mathcal{C}^\omega \to \mathcal{C}^\omega$. So we have an action of $\mathrm{Pic}(\mathcal{C})$ on $\mathcal{C}^\omega$. We define the open neighborhoods of the identity to be the stabilizers, i.e. for every $X \in \mathcal{C}^\omega$, we declare $\mathrm{Stab}(X) = \{M \in \mathrm{Pic}(\mathcal{C}) \mid M \otimes X \cong X\}$ to be open.

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