Algebraic K-Theory of Finite Fields

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Some useful resources on the topic, besides Quillen's paper [Qui72], are [Hai; Mes; Pé]. A modern account of Quillen's plus construction, which we follow, is given in [Nik17].

1 Introduction

The goal of this talk is to explain Quillen's +-construction, from a modern point of view, and to use it to compute the algebraic K-theory of finite fields $\mathbb{F}_q$ for q a power of p, following Quillen's paper, whose main result is

Theorem 1 (Quillen [Qui72])**.** *The algebraic K-theory of* $\mathbb{F}_q$ *is* $\mathrm{K}_0(\mathbb{F}_q) = \mathbb{Z}$, $\mathrm{K}_{2i}(\mathbb{F}_q) = 0$ *and* $\mathrm{K}_{2i-1}(\mathbb{F}_q) = \mathbb{Z}/\left(q^i - 1\right)$*.*

Recall that we defined $\mathrm{K}(R) = (\mathrm{Proj}_R^{\simeq})^{\mathrm{gpc}}$, that is we take the commutative monoid given by the groupoid of finitely-generated projective R-modules together with addition given by direct sum, and then group complete (i.e. apply the adjoint of the inclusion $\mathrm{Ab}(\mathcal{S}) \to \mathrm{CMon}(\mathcal{S})$). In the case where $R = F$ is a field, f.g. projective just means finite-dimensional, thus $\mathrm{Proj}_F^{\simeq} = \mathrm{Vect}_F^{\simeq} = \coprod \mathrm{BGL}_n(F)$. So $\mathrm{K}(F) = (\coprod \mathrm{BGL}_n(F))^{\mathrm{gpc}}$.

The main two ingredients will be Quillen's +-construction, which is a way to compute the group completion, and a comparison with $\mathrm{K}(\mathbb{C})$, which in turn is very related to *topological* K-theory, for which we have much more knowledge.

2 Quillen's +-Construction

The construction of algebraic K-theory involves taking group completion $M^{\mathrm{gpc}} \in \mathrm{Ab}(\mathcal{S})$. This is a complicated operation, and our first goal is to develop a computational tool to compute the underlying space $\underline{M^{\mathrm{gpc}}} \in \mathcal{S}$. Our first observation is

Proposition 2. *Let* $A \in \mathrm{Ab}(\mathcal{S})$*, then* $\pi_0 A$ *is an abelian group, and so is* $\pi_1(A, a)$ *for any point* $a \in A$*.*

Proof. $\pi_0 A$ has the abelian group structure coming from multiplication.

Note that all the connected components of A are equivalent: for any $a \in A$ the map $x \mapsto xa^{-1}$ is invertible, and restricts to an equivalence between the connected component of a and that of 1. In particular $\pi_1(A, a) \cong \pi_1(A, 1)$, so we focus on $\pi_1(A, 1)$. This has an extra group structure on top of the usual one (concatenation) that comes from the multiplication. These two distribute over each other, so by the Eckmann-Hilton argument, they are the same and abelian, thus $\pi_1(A, 1)$ is an abelian group as well. $\square$

Our approach to computing $\underline{M}^{\text{gpc}}$ is based on these two observations, that is we can try and "fix" π_0 and π_1 of M separately.

2.1 π_0 and the $(-)_\infty$-Construction

We begin by considering a *usual* commutative monoid $M \in \text{CMon}(\text{Set})$. The standard example is of course $\mathbb{N}^{\text{gpc}} = \mathbb{Z}$ obtained by inverting the element 1.

Definition 3. For $x \in M$, define $M_x = \operatorname{colim}\left(M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots\right)$. For a finite collection of elements $I \subseteq M$ define M_I by iterating the above colimit. Define $M_\infty = \operatorname{colim}_{I \subseteq M} M_I$ for all finite subsets $I \subseteq M$.

Proposition 4. $M_x = M\left[x^{-1}\right]$ *(i.e. maps from M_x are the same as maps from M with x mapping to invertible).*

Proof idea. We think about it as $\operatorname{colim}\left(M \to \frac{1}{x}M \to \frac{1}{x^2}M \to \cdots\right)$ i.e. write elements in the n-th place formally as $\frac{m}{x^n}$, and map $\frac{m}{x^n}$ to $\frac{mx}{x^{n+1}}$. $\square$

Corollary 5. *Let $M \in \text{CMon}(\text{Set})$, then $M_\infty = M^{\text{gpc}}$.*

We can try and mimic this in spaces, so let $M \in \text{CMon}(\mathcal{S})$.

Definition 6. For $x \in \pi_0 M$, define $M_x = \operatorname{colim}\left(M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \cdots\right)$. For a finite collection of elements $I \subseteq \pi_0 M$ define M_I by iterating the colimit. Lastly, define $M_\infty = \operatorname{colim}_{I \subseteq \pi_0 M} M\left[I^{-1}\right]$ over finite subsets $I \subseteq \pi_0 M$.

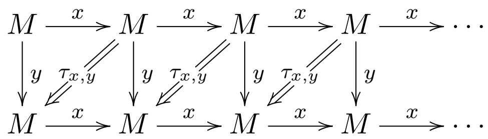
Proposition 7. $\pi_0(M_\infty) = (\pi_0 M)_\infty = (\pi_0 M)^{\text{gpc}}$*, in particular it is an abelian group.*

Proof. π_0 is a left adjoint so commutes with colimits. $\square$

Because of that, we can think of the operation $(-)_\infty$ as "fixing" π_0, and it is fairly concrete and easy to compute. By construction, there is a map $M_\infty \to M^{\text{gpc}}$, and one might hope that it is an equivalence, but this is *false* in general.

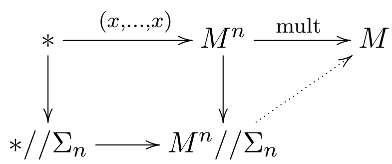
Example 8. Take $M = \coprod \mathrm{BGL}_n(F)$. We can compute M_∞ by using a set of generators of $\pi_0 M$. In this case $\pi_0 M = \mathbb{N}$, and a generator is a point $x \in \mathrm{BGL}_1(F)$. The operation here is given by going to the next connected components using (B of) $X \mapsto \begin{pmatrix} X & \\ & 1 \end{pmatrix}$. Taking the colimit we get $M_\infty = \mathbb{Z} \times \mathrm{BGL}_\infty(F)$. In particular, note that $\pi_1 M_\infty = \mathrm{GL}_\infty(F)$ (at any base-point) which is not abelian in general, so it can't be the case that $M_\infty = M^{\mathrm{gpc}}$. Furthermore, one can show that the action of x on M_x (which we define below) is taking to the next connected component using (B of) $X \mapsto \begin{pmatrix} 1 & \\ & X \end{pmatrix}$, which is not invertible.

Let's explain what goes wrong. It is important to understand where these colim's are taking place. As we just saw, in our case $\pi_0 M = \mathbb{N}$ so that $M_\infty = M_x$ for x a generator, and we focus on this for simplicity. The map $M \xrightarrow{x} M$ is *not* a map of commutative of monoids (it doesn't even send 1 to 1!), so we can't take the colim in $\mathrm{CMon}(\mathcal{S})$. It is a map of spaces with an M-action, so we can take the colimit there (whose underlying space is the colimit computed in spaces). For that we need to say how M acts on M_x (as this is extra structure). At the very least, for every $y \in M$ we need a map $M_x \xrightarrow{y} M_x$. Consider the following diagram:



Since this is done in spaces, to say that the diagram commute we need to fill each square by a homotopy. The homotopy is given in the commutativity structure of M by paths $\tau_{x,y} : yx \rightsquigarrow xy$. This diagram then induces a map on the colimits $M_x \xrightarrow{y} M_x$.

The question is then why isn't $M_x \xrightarrow{x} M_x$ necessarily invertible? Note that in its definition we use the path $\tau_{x,x} : xx \rightsquigarrow xx$. This path need not be the constant path! (A familiar example is for X, Y sets, $\tau_{X,Y} : X \coprod Y \to Y \coprod X$ is an equivalence, and when $Y = X$ we get the switch map which is not the identity.) Back to M, iterated multiplications yields the diagram



and commutativity is exhibited by the data of such a factorization. Taking π_1 we get $\Sigma_n \to \pi_1(M, x^n)$, which give obstructions to the invertibility of $M_x \xrightarrow{x} M_x$.

Theorem 9. *The following are equivalent:*

1. The canonical map $M_\infty \to M^{\mathrm{gpc}}$ is an equivalence.
2. $\pi_1 M_\infty$ is abelian (for all base-points).
3. $\pi_1 M_\infty$ is hypoabelian (for all base-points).
4. The cycle $(12\cdots n)$ is in the kernel of the composition $\Sigma_n \to \pi_1(M, x^n) \to \pi_1(M_\infty, x^n)$ for some $n \geq 2$.

Definition 10. A group P is called *perfect* if $P^{\mathrm{ab}} = 1$ (i.e. $[P,P] = P$). A group G has a maximal perfect subgroup (because if $P_1, P_2 \leq G$ are perfect, then $\langle P_1, P_2 \rangle \leq G$ is perfect as well), and it is called *hypoabelian* if the maximal perfect subgroup is trivial (equivalently, the derived series terminates at 1 transfinitely, so this can be thought of as transfinitely solvable).

Remark 11. Note for $n = 2$, the map $\Sigma_2 \to \pi_1(M_\infty, x^2)$ appearing theorem 9, sends (12) to τ_{xx}. So saying that it is in the kernel means that τ_{xx} is null-homotopic.

2.2 π_1 and the $(-)^+$-Construction

As we said, we have both a π_0 and a π_1 "problem", and our "fix" for π_0 didn't get us all the way to the group completion. The next step is to force π_1 to be hypoabelian.

Definition 12. Let $\mathcal{S}^{\mathrm{hypo}} \subseteq \mathcal{S}$ be the full subcategory of spaces such that $\pi_1(X, x)$ is hypoabelian for all base-points.

Theorem 13. *The inclusion $\mathcal{S}^{\mathrm{hypo}} \subseteq \mathcal{S}$ has a left adjoint called the plus construction $(-)^+ : \mathcal{S} \to \mathcal{S}^{\mathrm{hypo}}$.*

Proof. We show that $\mathcal{S}^{\mathrm{hypo}}$ is closed under limits. This follows from closure under (arbitrary) products which is immediate (since π_1 commutes with products and product of hypoabelian is hypoabelian), and closure under pullbacks. Let

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

be a pullback diagram where $X, Y, Z \in \mathcal{S}^{\mathrm{hypo}}$, and we want to show that $W \in \mathcal{S}^{\mathrm{hypo}}$ as well. We have a fibration $\Omega Z \to W \to X \times Y$, which leads to a long exact sequence of homotopy groups, $\cdots \to \pi_2(Z) \to \pi_1(W) \to \pi_1(X) \times \pi_1(Y) \to \cdots$, so $\pi_1(W)$ is an extension of a (hypo)abelian group and a hypoabelian group, therefore it is hypoabelian. □

Remark 14. The reason we work with hypoabelian rather than abelian, is that spaces with abelian π_1 are not closed under limits, so a left adjoint does not exist.

Proposition 15. *$X \to X^+$ is a homology equivalence.*

Proof. For any (usual) abelian group A of course $\mathrm{B}^n A \in \mathcal{S}^{\mathrm{hypo}}$ so by adjunction $\mathrm{Map}\left(X^+, \mathrm{B}^n A\right) = \mathrm{Map}\left(X, \mathrm{B}^n A\right)$ so that $\mathrm{H}^n\left(X^+; A\right) = \mathrm{H}^n\left(X; A\right)$. Using universal coefficient theorem, the result follows for homology. □

Proposition 16. *$(-)^+$ commutes with products, and in particular we also get $(-)^+ : \mathrm{CMon}\left(\mathcal{S}\right) \to \mathrm{CMon}\left(\mathcal{S}^{\mathrm{hypo}}\right)$.*

Proof. First, note that $\mathrm{Map}\left(X, Z\right) = \mathrm{Map}\left(\mathrm{colim}_X *, Z\right) = \lim_X Z$ so if $Z \in \mathcal{S}^{\mathrm{hypo}}$ then by closure under limits so is $\mathrm{Map}\left(X, Z\right)$.

Now, by Yoneda, for any $X, Y \in \mathcal{S}$ and $Z \in \mathcal{S}^{\mathrm{hypo}}$ we have

$$\begin{aligned}
\mathrm{Map}\left(\left(X \times Y\right)^+, Z\right) &= \mathrm{Map}\left(X \times Y, Z\right) \\
&= \mathrm{Map}\left(X, \mathrm{Map}\left(Y, Z\right)\right) \\
&= \mathrm{Map}\left(X^+, \mathrm{Map}\left(Y^+, Z\right)\right) \\
&= \mathrm{Map}\left(X^+ \times Y^+, Z\right)
\end{aligned}$$

□

Theorem 17. *Let $M \in \mathrm{CMon}\left(\mathcal{S}\right)$ then $\left(M_\infty\right)^+ = \left(M^+\right)_\infty = M^{\mathrm{gpc}}$ (as spaces).*

Proof. Recall that we have maps $M \to M_\infty \to M^{\mathrm{gpc}}$. Since M^{gpc} has (hypo)abelian π_1, by adjunction we get a commutative square

$$\begin{array}{ccc}
\left(M_\infty\right)^+ & \longrightarrow & M^{\mathrm{gpc}} \\
\downarrow & & \downarrow \\
\left(M^+\right)_\infty & \longrightarrow & \left(M^+\right)^{\mathrm{gpc}}
\end{array}$$

The bottom map is an equivalence, because the commutative monoid M^+ has hypoabelian π_1 so by theorem 9 $\left(M^+\right)_\infty = \left(M^+\right)^{\mathrm{gpc}}$.

The right map is an equivalence, because $\mathrm{Ab}\left(\mathcal{S}\right) \subseteq \mathrm{CMon}\left(\mathcal{S}^{\mathrm{hypo}}\right) \subseteq \mathrm{CMon}\left(\mathcal{S}\right)$ and so the adjoints are equivalent $\left(\left(-\right)^+\right)^{\mathrm{gpc}} = \left(-\right)^{\mathrm{gpc}}$.

We show that the left map is an equivalence using Yoneda. For simplicity assume again that π_0 is generated by a single element x, then for any Z

$$\begin{aligned}\mathrm{Map}\left((M_\infty)^+, Z\right) &= \mathrm{Map}\,(M_\infty, Z)\\ &= \mathrm{Map}\left(\mathrm{colim}\left(M \xrightarrow{x} M \xrightarrow{x} \cdots\right), Z\right)\\ &= \lim\left(\mathrm{Map}\,(M,Z) \xrightarrow{x} \mathrm{Map}\,(M,Z) \xrightarrow{x} \cdots\right)\\ &= \lim\left(\mathrm{Map}\left(M^+,Z\right) \xrightarrow{x} \mathrm{Map}\left(M^+,Z\right) \xrightarrow{x} \cdots\right)\\ &= \mathrm{Map}\left(\mathrm{colim}\left(M^+ \xrightarrow{x} M^+ \xrightarrow{x} \cdots\right), Z\right)\\ &= \mathrm{Map}\left(\left(M^+\right)_\infty, Z\right)\end{aligned}$$

And the top map is an equivalence because the three others are. □

3 Topological K-Theory

When we compute $\mathrm{K}(\mathbb{C}) = (\mathrm{Vect}_{\mathbb{C}}^{\simeq})^{\mathrm{gpc}}$ we consider $\mathrm{Vect}_{\mathbb{C}}^{\simeq} = \coprod \mathrm{BGL}_n(\mathbb{C})$. It is important to note that $\mathbb{C}$ is regarded as a *discrete* field, and so $\mathrm{GL}_n(\mathbb{C})$ is also a *discrete* group. We could do a topological version that takes into account the topology on $\mathbb{C}$, i.e. replace $\mathrm{Vect}_{\mathbb{C}}^{\simeq}$ by $\mathrm{Vect}_{\mathbb{C}}^{\simeq,\mathrm{top}} = \coprod \mathrm{BGL}_n^{\mathrm{top}}(\mathbb{C}) = \coprod \mathrm{BU}(n)$ (the spaces $\mathrm{BGL}_n^{\mathrm{top}}(\mathbb{C})$ and $\mathrm{BU}(n)$ are indeed equivalent, essentially by Gram-Schmidt), and we get

Definition 18. $\mathrm{K}^{\mathrm{top}}(\mathbb{C}) = \left(\mathrm{Vect}_{\mathbb{C}}^{\simeq,\mathrm{top}}\right)^{\mathrm{gpc}} = (\coprod \mathrm{BU}(n))^{\mathrm{gpc}}$.

Theorem 19. $\mathrm{K}^{\mathrm{top}}(\mathbb{C}) = \mathbb{Z} \times \mathrm{BU}$.

Proof. Note that $(\coprod \mathrm{BU}_n)_\infty = \mathbb{Z}\times\mathrm{BU}$ as we have seen before. Furthermore, $\pi_1(\mathbb{Z}\times\mathrm{BU}) = \pi_0\mathrm{U}$, so since U we get that $\pi_1 = 0$. In particular it is (hypo)abelian, so by theorem 9 we know that $(\coprod \mathrm{BU}(n))_\infty = (\coprod \mathrm{BU}(n))^{\mathrm{gpc}}$. □

$\mathrm{K}^{\mathrm{top}}(\mathbb{C}) = \mathbb{Z}\times\mathrm{BU}$ is known as connective topological K-theory (as a connective spectrum it is usually denoted by ku). This space classifies virtual (i.e. formal differences of) vector bundles up to stable isomorphism. Namely, $\mathrm{BU}(n)$ classifies rank n vector bundles, by their equivalence with $\mathrm{U}(n)$-principal bundles. Then one can show that $\mathrm{KU}(X) := \pi_0\mathrm{Map}(X, \mathbb{Z}\times\mathrm{BU})$ is the set of virtual vector bundles on X modulo $V \sim U$ iff $V \oplus \mathbb{C}^k \cong V \oplus \mathbb{C}^k$ for some k. This object was studied heavily, and in particular satisfies

Theorem 20 (Bott Periodicity)**.** $\pi_n(\mathbb{Z}\times\mathrm{BU}) = \mathrm{KU}(S^n) = \begin{cases}\mathbb{Z} & n = 2k\\ 0 & n = 2k-1\end{cases}$.

4 Proof of the Main Theorem

Recall that our goal was to compute $\mathrm{K}(\mathbb{F}_q)$. We will do this using the $+$-construction, i.e. we would like to understand $\mathbb{Z} \times \mathrm{BGL}_\infty(\mathbb{F}_q)^+$. We will do this by a comparison with $\mathrm{K}^{\mathrm{top}}(\mathbb{C})$, namely we will construct a map $\theta : \mathrm{BGL}_\infty(\mathbb{F}_q) \to \mathrm{BU}^{\psi^q}$, where BU^{ψ^q} is some fixed points of BU whose homotopy groups are easy to compute. Then we show that θ induces an isomorphism on $\mathrm{H}_*(-;\mathbb{Z})$, thus so does $\mathrm{BGL}_\infty(\mathbb{F}_q)^+ \to \mathrm{BU}^{\psi^q}$, from which we will conclude that it is a homotopy equivalence, and finish the proof.

The existence of θ may seem very odd at first. Doing this for all q is almost the same as mapping $\mathrm{GL}_n\left(\overline{\mathbb{F}}_p\right) \to \mathrm{U}(n)$, the first being a discrete group and the second a compact Lie group. These objects are not as far from each other as one may think. Consider the case $\mathrm{GL}_1\left(\overline{\mathbb{F}}_p\right) \to \mathrm{U}(1)$ i.e. $\overline{\mathbb{F}}_p^\times \to \mathrm{U}(1)$. We note that $\overline{\mathbb{F}}_p^\times = \operatorname{colim} \mathbb{F}_{p^k}^\times \overset{\text{non-canonically}}{\cong} \operatorname{colim} \mathbb{Z}/\left(p^k - 1\right) = \oplus_{\ell \neq p} \mathbb{Z}/\ell^\infty$, and $\mathbb{Z}/\ell^\infty$ canonically injects into $\mathrm{U}(1)$ as ℓ-th power roots of unity. Therefore, we get a (non-canonical) injection $\sigma : \overline{\mathbb{F}}_p^\times \hookrightarrow \mathrm{U}(1)$. This may still seem far off, but in fact $\mathrm{B}\mathbb{Z}/\ell^\infty \to \mathrm{BU}(1)$ is a homotopy equivalence after ℓ-adic completion though we will not use this fact. Our next step is to bootstrap σ to get θ.

4.1 Brauer Lifts

To fix notation, $\mathrm{Rep}_F(G)$ is the semi-ring of representations of G over F up to isomorphisms, and $R_F(G)$ is the representation ring ($= \mathrm{Rep}_F(G)^{\mathrm{gpc}}$). Quillen used a technique he called Brauer lifts to bootstrap the σ, building on the following theorem

Theorem 21 (Green [Gre55])**.** *Let G be a finite group and $\rho \in \mathrm{Rep}_{\overline{\mathbb{F}}_p}(G)$. Define $\chi_\rho : G \to \mathrm{U}(1)$ by $\chi_\rho(g) = \sum_{\lambda \in \mathrm{ev}\rho(g)} \sigma(\lambda)$ where the sum is over the eigenvalues (with multiplicity). Then χ is the character of a virtual complex representation of G.*

This allows us to lift representations from $\overline{\mathbb{F}}_p$ to $\mathbb{C}$, one we choose the map $\sigma : \overline{\mathbb{F}}_p^\times \hookrightarrow \mathrm{U}(1)$ from before, i.e. we get a map $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(G) \to R_{\mathbb{C}}(G)$ (which obviously preserves $\oplus, \otimes$ so it is a map of (semi-)rings).

We know that $R_{\mathbb{C}}(G) = \mathrm{Rep}_{\mathbb{C}}(G)^{\mathrm{gpc}} = \pi_0\left(\mathrm{Map}\left(\mathrm{B}G, \coprod \mathrm{BU}(n)\right)^{\mathrm{gpc}}\right)$. Furthermore, (by adjunction) there is a map $\mathrm{Map}\left(\mathrm{B}G, \coprod \mathrm{BU}(n)\right)^{\mathrm{gpc}} \to \mathrm{Map}\left(\mathrm{B}G, \left(\coprod \mathrm{BU}(n)\right)^{\mathrm{gpc}}\right)$, taking π_0 we get a map $R_{\mathbb{C}}(G) \to \mathrm{KU}(\mathrm{B}G)$ (known from the Atiyah-Segal theorem). Combining everything we get a map $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(G) \to R_{\mathbb{C}}(G) \to \mathrm{KU}(\mathrm{B}G)$.

4.2 Galois Action and Adams Operations

All of the representation (semi-)rings, character rings and topological K-theory, have extra structure: the exterior powers $\Lambda^k : R \to R$. In fact this endows R with the structure of λ-(semi-)ring, but we do not have time to give precise definitions. In a general ring, there is another way to encode the λ-structure by a collection of maps called the Adams operations $\psi^k : R \to R$. Although less familiar, they have various nice properties (which in fact uniquely characterize them in our cases):

1. $\psi^k : R \to R$ is a ring homomorphism.
2. For any "line element" (line-bundle or 1-dimensional representation) $\psi^k(L) = L^k$.
3. $\psi^k \psi^m = \psi^{km}$.

Proposition 22. *The action of* ψ^k *on characters of (virtual) representations is given by* $\left(\psi^k \chi\right)(g) = \chi\left(g^k\right)$.

Proposition 23. *For a representation* $\rho \in \mathrm{Rep}_{\mathbb{F}_q}(G) \subseteq \mathrm{Rep}_{\overline{\mathbb{F}}_p}(G)$*, the induced character satisfies* $\psi^q \chi_\rho = \chi_\rho$*, i.e.* χ_ρ *is a* ψ^q *fixed point.*

Proof. By definition, ρ is a fixed point of Frob^q so $\chi_\rho = \chi_{\mathrm{Frob}^q \rho}$. For a matrix A over $\overline{\mathbb{F}}_p$ with eigenvalues λ_i, the eigenvalues of $\mathrm{Frob}^q A$ are $\mathrm{Frob}^q(\lambda_i) = \lambda_i^q$, which are also the eigenvalues of A^q, so we get

$$\chi_\rho(g) = \chi_{\mathrm{Frob}^q \rho}(g) = \sum_{\lambda \in \mathrm{ev}(\mathrm{Frob}^q \rho(g))} \sigma(\lambda) = \sum_{\lambda \in \mathrm{ev}(\rho(g^q))} \sigma(\lambda) = \chi_\rho(g^q) = (\psi^q \chi_\rho)(g)$$

□

By a Yoneda argument, we can see that ψ^q acts on BU (representing KU). One can show that $\mathrm{KU}(\mathrm{B}G)^{\psi^q} = \pi_0 \mathrm{Map}(\mathrm{B}G, \mathbb{Z} \times \mathrm{BU})^{\psi^q} = \pi_0 \mathrm{Map}\left(\mathrm{B}G, \mathbb{Z} \times \mathrm{BU}^{\psi^q}\right)$ (using the fact that $\mathrm{KU}^1(\mathrm{B}G) = 0$, as follows from the Atiyah-Segal theorem).

Theorem 24. $\pi_{2n-1}\left(\mathrm{BU}^{\psi^q}\right) = \mathbb{Z}/(q^n - 1)$ *and* $\pi_{2n}\left(\mathrm{BU}^{\psi^q}\right) = 0$.

Proof. Recall from theorem 20 that $\pi_{2n}(\mathrm{BU}) = \mathbb{Z}$ (and vanishes in odd degrees), it is known that ψ^q acts on it by multiplication by q^n. The fixed points can be computed as the fiber $\mathrm{BU}^{\psi^q} \to \mathrm{BU} \xrightarrow{\psi^q - \mathrm{id}} \mathrm{BU}$. This give a LES in homotopy groups

$$0 \to \pi_{2n}\left(\mathrm{BU}^{\psi^q}\right) \to \mathbb{Z} \xrightarrow{q^n - 1} \mathbb{Z} \to \pi_{2n-1}\left(\mathrm{BU}^{\psi^q}\right) \to 0$$

and the result follows. □

Our map $\mathrm{Rep}_{\overline{\mathbb{F}}_p}(G) \to R_{\mathbb{C}}(G) \to \mathrm{KU}(\mathrm{B}G)$ is easily seen to preserve the exterior powers Λ^k, and therefore also preserve the Adams operations. Therefore, by the above, the restriction to $\mathrm{Rep}_{\mathbb{F}_q}(G)$ lands in $\mathrm{KU}(\mathrm{B}G)^{\psi^q}$. Combining all of this we get a map $\mathrm{Rep}_{\mathbb{F}_q}(G) \to \pi_0 \mathrm{Map}\left(\mathrm{B}G, \mathbb{Z} \times \mathrm{BU}^{\psi^q}\right)$.

4.3 The Map θ and Conclusion of the Proof

We can do a Yoneda-style argument, take $G = \mathrm{GL}_n(\mathbb{F}_q)$ and the regular representation $\mathbb{F}_q^n$. In this case we get a map $\mathrm{BGL}_n(\mathbb{F}_q) \to \mathbb{Z} \times \mathrm{BU}^{\psi^q}$, and since the source is connected we get $\theta_n : \mathrm{BGL}_n(\mathbb{F}_q) \to \mathrm{BU}^{\psi^q}$. These maps are compatible, and we get $\theta : \mathrm{BGL}_\infty(\mathbb{F}_q) \to \mathrm{BU}^{\psi^q}$.

Most of Quillen's paper is in fact dedicated to the following, which we do not have enough time to explain.

Theorem 25. *The map $\theta : \mathrm{BGL}_\infty(\mathbb{F}_q) \to \mathrm{BU}^{\psi^q}$ induces an isomorphism on $\mathrm{H}_*(-;\mathbb{Z})$.*

Proof (very rough outline). By the universal coefficient theorem it is enough to check that it is an isomorphism with coefficients $\mathbb{Q}$, $\mathbb{Z}/p$ and $\mathbb{Z}/\ell$ for $\ell \neq p$.

By the computation of $\pi_*\left(\mathrm{BU}^{\psi^q}\right)$ we see that it vanishes rationally and $\mod p$, thus we need to prove the same for $\mathrm{BGL}_\infty(\mathbb{F}_q)$.

It is a well known and easy result that for a finite group G, $\tilde{\mathrm{H}}_*(\mathrm{B}G;\mathbb{Q}) = 0$. Then we get $\tilde{\mathrm{H}}_*(\mathrm{BGL}_\infty(\mathbb{F}_q);\mathbb{Q}) = \tilde{\mathrm{H}}_*(\operatorname{colim}\mathrm{BGL}_n(\mathbb{F}_q);\mathbb{Q}) = \operatorname{colim}\tilde{\mathrm{H}}_*(\mathrm{BGL}_n(\mathbb{F}_q);\mathbb{Q}) = 0$.

To prove that $\tilde{\mathrm{H}}_*(\mathrm{BGL}_n(\mathbb{F}_q);\mathbb{Z}/p) = 0$, one first uses a comparison with $n \times n$ upper triangular matrices, inductively on n, from which it follows for $* < \nu(p-1)$ where $q = p^\nu$. Then, using a transfer argument, i.e. passing to bigger fields $\mathbb{F}_{q^r}$, one gets the result for all $*$.

The hardest part is then to show that θ induces an isomorphism on $\mathrm{H}_*(-;\mathbb{Z}/\ell)$ for $\ell \neq p$. The first part concerns BU^{ψ^q}. Quillen applies the pullback ("Eilenberg-Moore") spectral sequence to

$$\begin{array}{ccc} \mathrm{BU}^{\psi^q} & \longrightarrow & \mathrm{BU}^{[0,1]} \\ \downarrow & & \downarrow \\ \mathrm{BU} & \xrightarrow{(\mathrm{id},\psi^q)} & \mathrm{BU} \times \mathrm{BU} \end{array}$$

the cohomology of BU is well known, and the spectral sequence collapses at the E_2-page, which gives some of the information about $\mathrm{H}^*\left(\mathrm{BU}^{\psi^q};\mathbb{Z}/\ell\right)$. The next ingredient is the group $C = \mathbb{F}_q(\zeta_\ell)^*$, and its $\mathbb{F}_q$ representation $\mathbb{F}_q(\zeta_\ell)$ (and powers of them). This has Brauer lift that can be described explicitly. It is encoded by $\mathrm{B}C \to \mathrm{BGL}_n(\mathbb{F}_q) \to \mathrm{BU}^{\psi^q}$ (for suitable n), and the induced map on cohomology $\mathrm{H}^*\left(\mathrm{BU}^{\psi^q};\mathbb{Z}/\ell\right) \to \mathrm{H}^*(\mathrm{B}C;\mathbb{Z}/\ell)$ gives enough information to finish the computation of $\mathrm{H}^*\left(\mathrm{BU}^{\psi^q};\mathbb{Z}/\ell\right)$ and its homology. Lastly, using the factorization through $\mathrm{BGL}_n(\mathbb{F}_q)$, Quillen imports the data from both of these to the homology of $\mathrm{BGL}_n(\mathbb{F}_q)$, showing the desired isomorphism. □

Recall from 15 that the +-construction is a homology equivalence, therefore we get that $\mathrm{K}(\mathbb{F}_q) = (\mathbb{Z} \times \mathrm{BGL}_\infty(\mathbb{F}_q))^+ \to \mathbb{Z} \times \mathrm{BU}^{\psi^q}$ incudes an isomorphism on $\mathrm{H}_*(-;\mathbb{Z})$ as well.

Theorem 26 (Emmanuel Farjoun [Dro71]). *Let $f : X \to Y$ be a map of simple spaces (abelian π_1, acting trivially on π_n). If f induces an isomorphism on $\mathrm{H}_*(-;\mathbb{Z})$, then it is also a homotopy equivalence.*

This is a generalization of the standard Whitehead theorem. It is known that the underlying space of any object of $\mathrm{Ab}(\mathcal{S})$ is simple, so $\mathrm{K}(\mathbb{F}_q)$ is simple. Furthermore, the fibration used to compute the homotopy groups of BU^{ψ^q} also shows that it is simple. Therefore, $\mathrm{K}(\mathbb{F}_q) \to \mathbb{Z} \times \mathrm{BU}^{\psi^q}$ is an equivalence, and the main result follows

Theorem 27 (Quillen [Qui72]). *The algebraic K-theory of $\mathbb{F}_q$ is $\mathrm{K}_0(\mathbb{F}_q) = \mathbb{Z}$, $\mathrm{K}_{2i}(\mathbb{F}_q) = 0$ and $\mathrm{K}_{2i-1}(\mathbb{F}_q) = \mathbb{Z}/(q^i - 1)$.*

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