Milnor's λ invariant and exotic 7-spheres

November 18, 2019

Introduction:

Classifying manifold, in different manifolds categories, is a central problem in geometry. Therefore a natural question is how those isomorphism classes differ e.g Given a manifold M are

$$\{N \mid N \text{ } homotopic \text{ } to \text{ } M\} \quad \{N \mid N \text{ } homeomorphic \text{ } to \text{ } M\} \quad \{N \mid N \text{ } diffomorphic \text{ } to \text{ } M\}$$

the same. For $M = S^7$ we know from the general Poincare' conjecture that the first and second sets are the same. our main goal today is to show, constructively, that the third and second sets are not.

Our first short term goal will be to contract a smooth structure invariant. But in order to do so we will need to take a short detour and talk a bit about characteristic classes.

Characteristic Classes:

Recall that a G bundle is a fiber bundle $F \to E \xrightarrow{p} B$ such that G acts on F and $h_i^{-1} \circ h_j(x, v) = (x, g_{ij}(x)v)$ were $g_{ij} : U_i \cap U_j \to G$ and h_i are the trivialization maps.

Definition. A characteristic classes of degree i is a natural transformation:

$$\alpha_{(-)} : Vect_n(-) \to H^i(-)$$

were $Vect_n(B)$ is the set of isomorphism classes of n-bundles over B.

Let ϵ^k be the trivial k-bundle over B. Then ϵ^k is the pull-back of the bundle $\mathbb{R}^k \to pt$ so from naturally $\alpha_B(\epsilon^k) = 0$.

Let $E \xrightarrow{p} B$ be a vector bundle, we denote by E_0 the set of all non zero vectors in E and by E_x the fiber over $x \in B$. Recall that an oriented n-vector bundle is a vector bundle with a structure group $SL(n)$ and that the preferred orientation on E_x is the pull-back via the trivialization map of the regular orientation on

$\mathbb{R}^n$ i.e the pull-back of the preferred generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. We will need the following definitions and Theorems.

Theorem. *(Thom isomorphism) Let E be an oriented vector bundle. Then there is a unique $u \in H^n(E, E_0)$ such that $\forall x \in B$*

$$u_{|(E_x, E_x - \{0\})} = u_x \in H^n(E_x, E_x - \{0\}).$$

furthermore $H^*(E, E_0)$ is a $H^*(E)$- module, and the map $y \overset{\varphi}{\mapsto} y \cdot u$ is an isomorphism between $H^j(E)$ and $H^{n+j}(E, E_0)$

Note that E and B are homotopicly equivalent since E retracts to B.

Definition. Let $E \overset{p}{\to} B$ be a n-bundle and $i : E \to (E, E_0)$ be the embedding. We define the euler class, $e(E)$, by $p^* i^*(u)$

Note that from the uniqueness of u, $e(-)$ is characteristic class.

Corollary. *the following is a long exact sequence:*

$$\to H^i(B;\mathbb{Z}) \xrightarrow{\cup e} H^{i+n}(B;\mathbb{Z}) \xrightarrow{p_0^*} H^{i+n}(E_0;\mathbb{Z}) \overset{\varphi^{-1}\delta}{\to} H^{i+1}(B;\mathbb{Z}) \xrightarrow{\cup e}$$

were δ is the connecting morphism in the long exact sequence of pairs. It is called the Gysin long exact sequence.

Proof. the following digram is commutative:

$$\begin{array}{ccccccc} H^{n+j}(E, E_0) & \xrightarrow{i^*} & H^{n+j}(E) & \longrightarrow & H^{n+j}(E_0) & \xrightarrow{\delta} & H^{n+j+1}(E, E_0) \\ \varphi\uparrow & & p^*\uparrow & & Id\uparrow & & \uparrow\varphi \\ H^j(B) & \xrightarrow{\cup e} & H^{n+j}(B) & \xrightarrow{p_0^*} & H^{n+j}(E_0) & \xrightarrow{\varphi^{-1}\delta} & H^{j+1}(B) \end{array}$$

□

Theorem. *Fix a an orientation on BU then $H^*(BU) = \mathbb{Z}[1, c_1, c_2, ...]$ $|c_i| = 2i$ and c_i are uniquely determined. A change in orientation change the c_i by the formula $c_i = (-1)^i c_i'$.*

We fix a complex structure (an orientation) on $BU(n)$.

Definition. Let E be a complex n- bundle. Let $f : B \to BU(n)$ be the unique (up to homotopy) map such that $f^* EU(n) = E$. We define the ith' chern classes of E by the pull-back $f^*(c_i) =: c_i(E)$

Since $E \cong f^* EU(n)$ we can think on $c_i(E)$ as a way to measure how the fibre are linked. We also note that by definition $c_i(-)$ is a characteristic class.

We denote the total chern class by $c(E) := 1 + c_1(E) + ...$ and for a smooth manifold M we denote $c(M) := c(TM)$

Proposition. *Let E, E' be two vector bundle over B $c(E \oplus E) = c(E)c(E')$.(we omit the long and quite computational proof)*

Given a real n-bundle $E \xrightarrow{p} B$, we can construct its coplexfication $E \otimes_{\mathbb{R}} \mathbb{C}$ as the complex n-bundle over B with transition function given by the transition functions of E under the inclusion $GL(n;\mathbb{R}) \hookrightarrow GL(n;\mathbb{C})$.

There are canonical isomorphisms of complex vector bundle $E \otimes_{\mathbb{R}} \mathbb{C} = (E \otimes_{\mathbb{R}} \mathbb{C})^*(u + iv \mapsto u - iv)$ and, When E is complex $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E}$ $(x \otimes c \mapsto (cx, \bar{c}x))$. Note that from the first isomorphism $c_i(E \otimes_{\mathbb{R}} \mathbb{C}) = (-1)^i c_i(E \otimes_{\mathbb{R}} \mathbb{C})$ for i odd, i.e $2c_{2i-1}(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$.

Definition. we define the i-th Pontrjagin class of the real vector bundle $E \to B$ as $p_i(E) = (-1)^i c_{2i}(E \otimes_{\mathbb{R}} \mathbb{C})$.

Note that we get for free (from the property of chern classes) that $p(-)$ is a characteristic class and $2(p(E \oplus E') - p(E)p(E') = 0$ and $p(E \oplus \epsilon^k) = p(E)$.

We know calculate $p(S^n)$ (a result we will need later). Since the unit normal is a global section to the normal bundle $N := NS^n$, N is a trivial bundle. So $p(TS^n) = p(TS^n \oplus N) = 1$

The signature of a manifold:

Let M be a smooth compact oriented $4n$-manifold. We can define a symmetric bilinear form:

$$H^{2n}(M) \times H^{2n}(M) \to \mathbb{Z} \quad (x, y) \mapsto \langle [M], x \cup y \rangle$$

The signature of such a form is called the signature of M and denoted by $\sigma(M)$.

Theorem. *(Hirzebruch's signature theorem) there exist a polynomial L_n, such that for all smooth , oriented, compact manifold of dimension $4n$ the signature is given by*

$$\sigma(M) = \langle [M], L_n(p_1(M), ..., p_n(M)) \rangle$$

Proof. (outline)the polynomial L_n are called the Hirzebruch polynomial and have an explicate formula (that we won't give here for lack of time). Therefore One can check that both sides are cobordism invariant so it's suffice to check equality on generator of the cobordisem ring i.e on $\mathbb{PC}^n$ □

in particular L_n is homogeneous polynomial in the graded ring $\mathbb{Q}[x_1, ..., x_n]$ $|x_i| = i$ for example:

$$L_1 = \frac{1}{3}x_1 \quad L_2 = \frac{1}{45}(7x_2 + x_1^2)$$

Denote by c_n the coefficient of x_n in L_n. Define $L'_n(p_1, ...p_{n-1}) = L_n(p_1, ..., p_n) - c_n p_n$

Milnor's λ invariant:

We can now finely define Milnor's λ invariant. Let M be a $4n - 1$ manifold that has the same homology as

S^{4n-1} (this is equvelant to being a homotopy sphere). Suppose that we are given a $4n$ dimensional manifold B such that $\partial B = M$ we define:

Definition.

$$\lambda(M) = \frac{1}{c_n}\left(\sigma(B) - \langle [B] L'\left(p_1, \ldots, p_{n-1}\right)\rangle\right) \in \mathbb{Q}/\mathbb{Z}$$

were for a manifold with boundary we take the relative analogue of σ, i.e the signature of the map $H^{2n}(B, M) \times H^{2n}(b, M) \to \mathbb{Z}$

<u>λ is well defined:</u> Let B_1, B_2 be two manifold with boundary M. Define $C = B_1 \cup_M B_2$ this becomes a compact, oriented manifold (without boundary) if we orient B_2 the opposite way. From the relative Mayer-Vietoris sequence gives isomorphism

$$h^* : H^{2n}(C, M) \to H^{2n}\left(B_1, M\right) \oplus H^{2n}\left(B_2, M\right)$$

From our assumption on M $j : H^{2n}(C, M) \to H^{2n}(C)$ is an isomorphism. Thus every $\alpha \in H^{2n}(C)$ is of the form $j^* {h^*}^{-1}(\alpha_1, \alpha_2) = \alpha$ and we get

$$\left\langle [C], \alpha^2 \right\rangle = \left\langle [C], j^* \left(h^*\right)^{-1}\left(\alpha_1^2, \alpha_2^2\right)\right\rangle = \left\langle j_*[C], \left(h^*\right)^{-1}\left(\alpha_1^2, \alpha_2^2\right)\right\rangle = \left\langle [B_1], \alpha_1^2\right\rangle - \left\langle [B_2], \alpha_2^2\right\rangle$$

We conclude

$$\sigma(C) = \sigma\left(B_1\right) - \sigma\left(B_2\right)$$

Note that the map $i_j : B_j \to C$ are embeddings so $TB_j = i_j^*(TC)$. So Mayer-Vietoris gives an isomorphism

$$h^* : H^{2k}(C) \to H^{2k}(B_1) \oplus H^2(B_2) \quad k \leq n-1$$

by our assumption so like before we get that

$$\langle [C], L'\left(p_1(C), \ldots, p_{n-1}(C)\right)\rangle = \langle [B_1], L'\left(p_1(B_1), \ldots, p_{n-1}(B_1)\right)\rangle - \langle [B_2], L'\left(p_1(B_2), \ldots, p_{n-1}(B_2)\right)\rangle$$

all and all

$$\frac{1}{c_n}\sigma(B_1) - \langle [B_1], L'\left(p_1, \ldots, p_{k-1}\right)\rangle - \left(\sigma(B_2) - \langle [B_2], L'\left(p_1, \ldots, p_{k-1}\right)\rangle\right) = \frac{1}{c_n}\left(\sigma(C) - \langle [C], L'\left(p_1, \ldots, p_{k-1}\right)\rangle\right) = 0 \in \mathbb{Q}/\mathbb{Z}$$

.

For us it would be more convenient to work mod 7, i.e

$$\lambda(M) = 2\langle [B], p_1(B)^2\rangle - \sigma(B) \quad mod(7)$$

this is well define by the same proof.

We got our invariance so now we need to build candidates for exotic spheres.

Fiber bundles over spheres:

the main idea here is to generalize the construction of quaternionic Hopf fibration. By that we mean taking sphere bundle of oriented $\mathbb{H}$ vector bundle over $S^4 \cong \mathbb{HP}^1$. (this is the Hopf fibration were E is the tautological bundle $\{(x,v) \in \mathbb{HP}^n \times \mathbb{H}^{n+1} \text{ s.t. } [v] = x\}$)

Let $U_1, U_2 \subset S^4$ be everything but the south and north pole respectively. So $U_i \cong \mathbb{H}$ and thus (since $\mathbb{H} \cong \mathbb{R}^4$ is contactable) every $SO(n)$ $\mathbb{H}$-bundle is of the form

$$\mathbb{H} \times \mathbb{H} \cup_f \mathbb{H} \times \mathbb{H}$$

were $f : U_1 \cap U_2 = S^3 \to SO(n)$ is the clutching function i.e $(x,y) = (x^{-1}, f(x)y) \in \mathbb{H} \times \mathbb{H} \cup_f \mathbb{H} \times \mathbb{H}$

In fact from loop suspension adjunction, $[\Sigma S^{n-1}, BSO(n)] \cong [S^{n-1}, \Omega BSO(n) = SO(n)] = \pi_{n-1}(SO(n))$, those are all the $\mathbb{H}$ bundle.Therefore we need to compute $\pi_3(SO(4))$

Proposition. $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$

Proof. Define

$$f : S^3 \times S^3 \to SO(4) \quad f(u,w)v = uvw$$

this is a group homomorphism with kernel $\{(1,1)(-1,-1)\}$ and thus a two cover so we get that $\pi_3(S^3 \times S^3) = \mathbb{Z} \oplus \mathbb{Z} \cong \pi_3(SO(4))$ and the isomorphism is given explicitly by

$$(i,j) \mapsto f'_{i,j}(u) = (u^i, u^j) \mapsto f_{i,j}(u)(v) = u^i v u^j$$

□

We denote the vector bundle that corresponds to (i,j) by $E_{ij} = \mathbb{H} \times \mathbb{H} \cup_{f_{ij}} \mathbb{H} \times \mathbb{H}$ and by $M_{ij} = \mathbb{H} \times S^3 \cup_{\bar{f_{ij}}} \mathbb{H} \times S^3$ the corresponding sphere bundle. Note that E_{01} is the tautological bundle

Computing λ:

Proposition. $p_1(E_{hj}) = 2(h-j)$
$e(E_{hj}) = (h+j)$

Proof. The computation of the Pontrjagin class is achieved in 3 steps:

1. $p_1(E_{hj})$ is linear in j and h: this is just the composition

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\sim} \pi_3(SO(4)) \to \pi_4(BSO(4)) \to H^4(S^4)$$

were the second map is from loop suspension adjunction and the third map is $[f] \mapsto p_1(f^* ESO(4))$.

2. $p_1(E_{hj}) = c(h-j)$ for some c: by conjugating the defining map $u \mapsto (v \mapsto u^i v u^j)$ we get an orientation reversing isomorphism $E_{ij} \cong E_{-j-i}$ so $p_1(E_{ij}) = p_1(E_{-j-i})$ and from linearity we are done.

3. $c = 2$: recall that E_{01} is the tautological vector bundle over $\mathbb{HP}^1$ in particular it is complex so:

$$p_1(E_{01_{\mathbb{R}}}) = -c_2(E_{01_{\mathbb{C}}} \otimes \mathbb{C}) = -c_2(E_{01_{\mathbb{C}}} \oplus \bar{E_{01_{\mathbb{C}}}}) = -[c_0(E_{01_{\mathbb{C}}})c_2(E_{01_{\mathbb{C}}}) + c_2(E_{01_{\mathbb{C}}})c_0(E_{01_{\mathbb{C}}})] = -2e$$

Similarly, the Euler class is linear in h and j, and, since reversing orientation carries such class to the negatve of itself, $e(E_{hj}) = c(h+j)$ for some c. Again, by comparison with the Euler class of the tautological bundle, we get $c = 1$ □

Since M_{ij} is homotopic equivalent to E_{ij_0} we can compute its cohomology via the Gysin sequence. Obviously $H^r(M_{hj}) = 0$ for every $r \neq 0, 3, 4, 7$ and $H^0(M_{hj}) = H^7(M_{hj}) = \mathbb{Z}$. For H^3 and H^4 by the Gysin sequence:

$$0 \to H^3(M) \to H^0\left(S^4\right) \overset{\cup e}{\to} H^4\left(S^4\right) \to H^4(M) \to 0$$

there for since $e(E_{hj}) = (h+j)$:

$$H^3\left(M\right) \cong \begin{cases} 0 & \text{if } h+j \neq 0 \\ \mathbb{Z} & \text{if } h+j = 0 \end{cases} \qquad H^4(M) \cong \frac{\mathbb{Z}}{(h+j)\mathbb{Z}}$$

so M_k were $k = j-h, j+h = 1$ is a homological sphere. Moreover, M_k is the boundary of the 8-manifold B_k, the total space of the unit disk bundle in E_{hj}; therefore we can compute $\lambda(M_k)$.

Theorem. $\lambda\left(M_k\right) \equiv k^2 - 1 (\mathrm{mod} 7)$

Proof. Since the inclusion of the zero section in B_k is a homotopy equivalence, it induces an isomorphism in cohomology $H^4(S^4) \cong H^4(B_k)$, so $H^4(B_k)$ is cyclic and $\sigma(Bk)$ can only be $+-1$.
Now the tangent bundle TB_k splits into the "vertical" piece (just the pull-back of E_{ij}) and the "horizontal" piece (the pull-back of TS^4). In other words, if $p: B_k \to S^4$ is projection, we have

$$TE_{hj} \cong p^*\left(TS^4\right) \oplus p^*\left(E_{hj}\right)$$

Therefore

$$p_1\left(TE_{hj}\right) = p^*\left(p_1\left(TS^4\right)\right) + p^*\left(p_1\left(E_{hj}\right)\right) = p^*(2k) = 2k\alpha$$

Were α is pull-back of the generator of $H^4(S^4)$ via $B_k \to E_{hj} \to S^4$. we conclude $\lambda(M) = 2\langle [B], (2k\alpha)^2 \rangle - 1 = k^2 - 1 \quad mod(7)$ □

note that for the regular sphere taking $B = D^7$ we get that $\lambda(S^7) = 0$. So we found homological spheres that are not diffomorphic to S^7.

M_k **is homeomorphic to** S^7**:**

Morse Theory is a way of translating the topology of a manifold into statements about critical points of particular functions. Therefore, the primary diffculty of applying Morse Theory lies in finding a function with a set of critical points that is easy to study. In particular, we want a minimal set of critical points with non singular Hessians.

Theorem. *given a smooth compact manifold M and a Morse function (non singular Hessians at critical points) $f : M \to \mathbb{R}$ denote the set of critical points by $a_0, ..., a_n$ then M has a CW structure with a j cell for every i such that the number of positive egiunvalues of $Hess_{a_i}(f)$ is j.*

Corollary. *Let M^n be a smooth compact manifold and $f : M \to \mathbb{R}$ be a Morse function with to critcal point then M is homeomorphic to S^n.*

for $M = \mathbb{H} \times S^3 \cup_{\bar{f_{ij}}} \mathbb{H} \times S^3$ $((u, v) \sim (u', v') = \left(u^{-1}, \frac{u^h v u^{1-h}}{\|u\|}\right), h = \frac{1+k}{2})$ we define in the rst chart:

$$f(u, v) = \frac{\Re(v)}{(1 + \|u\|^2)^{1/2}}$$

Which has critical point only at $(0, + - 1)$ (explain by drawing - this is just a projection normalize by the distance on the sphere). in the second chart:

$$f(u', v') = \frac{\Re\left((u')^h v' (u')^{1-h}\right)}{\left(1 + \|u'\|^2\right)^{1/2}}$$

which does not have a critical point. By changing coordinates (u'', v') (instead of (u', v')) with $u'' = v'u'$ we have

$$f(u''; v') = \frac{\Re(u'')}{(1 + |u''|^2)}$$

This means that f can be extended smoothly to the second chart, giving a smooth function on all M_k. Thus M_k is homeomorphic to S^7 .