Historical Motivation

(Not meant to be written down only to be said out loud to get everyone on board - most of this section will probably be skipped)
Algebraic Topology, in broad storkes, is the study of algebraic invariants of topological spaces (for the sake of concreteness here's a list of examples of spaces we'd like to fit in our theory: Smooth/Topological manifolds, Simplicial Complexes, CW Complexes, Algebraic Varieties etc...).
In the first half of the 20th-century many different constructions were discovered for attaching invariants to each of the above examples (De-Rham cohomology, Simplicial Homology, Cellular Homology, Singular Homology etc...).
Gradually it was realized that many of these invariants (although not all of them defined on the same class of topological spaces) were giving the same answers. Proving comparisons between the different constructions on overlapping domains of definition quickly became painfully technical. To solve this problem Eilenberg and Steenrod developed an axiomatic approach to the study of these invariants.

Official Start

Definition 1. *Let $Ho = Ho(Top)$ be the homotopy category of topological spaces, i.e. the localization of the category of topological spaces w.r.t. weak homotopy equivalences. This category is also equivalent $Ho(CW)$ whose objects are CW-complexes and morphisms homotopy classes of maps (this is how we will think about it for the rest of the talk).*

All of the examples mentioned before fit into this category obviously. Also all the invariants mentioned, when defined, are functorial and homotopy invariant. Here are the eilenberg-steenrod axioms for cohomology theories:

Definition 2. *A (reduced) classical* cohomology theory H^* is a functor*

$$H^* : Ho_*^{op} \to grAb$$

Equipped with natural isomorphisms:

$$H^{*+1}(\Sigma X) \cong H^*(X)$$

For every $X \in Ho_*$ (where Σ stands for reduced suspension). Satisfying the following conditions:

1. *(**Additive**) : The canonical map $H^*(\bigvee_\alpha X_\alpha) \to \prod_\alpha H^*(X_\alpha)$ is an isomorphism.*
2. *(**Exactness**) : For every inclusion $A \subset X$ of a subcomplex the associated sequence*

$$H^*(X/A) \to H^*(X) \to H^*(A)$$

   is exact in the middle.

3. *(**Dimension**) : $H^{\neq 0}(S^0) = 0$*

Cohomology theories form a category where morphisms are natural transformation respecting the suspension isomorphisms.

Proposition 1. *A morphism of cohomology theories inducing isomorphism on $H^0(S^0)$ is an isomorphism.*

Proof. By compatibility with the natural isomorphisms it induces iso on all spheres. Then by (1) it induces iso on all wedges of spheres. Then since every space has skeletal filtration whose quotients are wedges of spheres we can induct on the filtration using (2) for the induction step to win. $\square$

In fact a stronger statement is true in this case. The cohomology theory is in fact uniquely determined by its coefficient group. This can be shown by inductively constructing the natural transformation then using the above proposition (note that if the dimension axiom is dropped this becomes impossible!).
So far we have shown uniqueness but not existence. There's a way to prove existence by simply using cellular homology (and cellular approximation for maps). We will go in a different route. Let Ho_*^c denote the homotopy category of pointed, connected CW-complexs.

Theorem 1 (Brown's representability)**.** *A functor $F : Ho_*^c \to Set_*$ is representable precisely when it satisfies the following conditions:*

1. *Respects coproducts - for every collection $\{X_\alpha\}$ the natural map $F(\bigvee_\alpha X_\alpha) \to \prod_\alpha F(X_\alpha)$ is a bijection.*
2. *Mayer Vietoris - Whenever $X = A_1 \cup A_2$ (where A_1, A_2 are subcomplexes) the canonical map*
$$F(X) \to F(A_1) \times_{F(A_1 \cap A_2)} F(A_2)$$
is surjective (we will not prove this theorem).

Example 1. *Let H^n be n-graded piece of some cohomology theory. Condition (1) of Brown's theorem is $ES1$ and so immediately satisfied. The second conditions follows from applying $ES2$ and some diagram chasing on the morphism of exact sequences*

$$\begin{tikzcd}
H^n(X/A_1 \cap A_2) \arrow[r] \arrow[d] & H^n(X) \arrow[r] \arrow[d] & H^n(A_1 \cap A_2) \arrow[r] \arrow[d] & H^{n+1}(X/A_1 \cap A_2) \arrow[d] \\
H^n(\frac{A_1 \bigvee A_2}{A_1 \cap A_2}) \arrow[r] & H^n(A_1) \oplus H^n(A_2) \arrow[r] & H^n(A_1 \cap A_2)^{\oplus 2} \arrow[r] & H^{n+1}(\frac{A_1 \bigvee A_2}{A_1 \cap A_2})
\end{tikzcd}$$

The leftmost and rightmost arrows are isomorphisms abnd the second from the right is the diagonal map $x \mapsto (x, x)$.

Thereofore for any abelian group A there's a unique pointed space which we will denote by $K(A, n)$ satisfying that $[-, K(A, n)] = H^n(-, A)$. This space is called an eilenberg maclane space.

Remark 1. *Notice that by eilenberg steenrod axioms we must have $[S^n, K(A, n)] = H^n(S^n, A) \cong H^0(S^0, A) = A$ and the rest of the homotopy groups are trivial by the dimension axiom. It follows immediately from obstruction theory that $K(A, n)$ can also be characterized as the unique pointed space with these homotopy groups.*

Example 2. *The circle $S^1 = \mathbb{R}/\mathbb{Z}$ has fundamental group $\mathbb{Z}$ and no higher homotopy groups and thus is a $K(\mathbb{Z}, 1)$. Real Projective space $\mathbb{R}P^n$ has S^n as its universal 2-cover. Combining with $S^\infty \cong *$ we conclude that $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$.*

The above examples motivates the following construction for a group G.

Example 3. *Take contractible space EG with a free action of G then take the quotient $BG = EG/G$. By the long exact sequence of homotopy groups for $G \to EG \to BG$ we see that if G is abelian we get $BG = K(G, 1)$.*

Notice that unlike before this construction makes sense for arbitrary topological and/or non-abelian groups as well! (while $K(G,1)$ makes no sense at all in this case). Notice also that mapping this fibration with the path space fibration for BG gives an equivalence $\Omega BG = G$.

Remark 2. *Recall that if G is a topological group, a principal G-bundle over X is a space P with an free action of G s.t. $P/G = X$. Now note that the universal fibration $EG \to BG$ has 2-notable properties: EG is a free G-space and $EG/G = BG$. It turns out that both of these properties are preserved by pullbacks! (follows from the fact that colimits are universal and stabilizers can only decrease upon pullback). Thus any map $X \to BG$ can be used to pullback EG and obtain a principal G-bundle P. Therefore the space BG is the classifying space for principal G-bundles.*

Example 4. *Complex projective space $\mathbb{C}P^n$ can be defined as the quotient of $\mathbb{C}P^{n+1}$ by the scaling action of $\mathbb{C}^\times$ Restricting to the corresponding sphere we obtain a fiber sequence $U(1) \to S^{2n+1} \to \mathbb{C}P^n$. In the limit this gives us $\mathbb{C}P^\infty = BU(1)$ (we can thus conclude that $H^*(BU(1), Z) = \mathbb{Z}[[x]]$(because there are only even cells)) cool!.*

Notice that $BU(1) = BS1 = B(B\mathbb{Z}) = B^2\mathbb{Z}$ (in particular $\pi_2(BU(1)) = Z$ and is the only non-trivial homotopy group!) but what does that mean?

Remark 3. *Turns out that if A is a topological abelian group then BA can also be endowed with an abelian group structure (unique upto homotopy) as well. We can thus define for a discrete abelian group a topological abelian group B^nA recursively. By the long exact sequence in homotopy groups for $B^nA \to E(B^{n+1}A) \to B^{n+1}A$ we conclude that $B^nA = K(A,n)$.*

Example 5. *Let $U(n)$ be the unitary group has a classifying space $BU(n)$. Recall that it classifies principal $U(n)$ bundle but there's a correspondence between vector bundles and principal bundles which in one way takes the associated bundle and in the other the frame bundle (give details if there's time). So we can think of $BU(n)$ also as the classifying space of complex vector bundles of rank n. Also we define $BU := colim_n BU(n)$ which classifies virtual vector bundles (in a sense that will maybe become clear later in the day... *Thom spectra*). The cohomologies of these spaces be computed using the the serre spectral sequence for $S^{2n+1} \to BU(n) \to BU(n+1)$)*

Fix an abelian group A and recall that $\Omega B^{n+1}A = B^nA$. We thus have an infinite sequence of spaces each delooping the next. We are motivated to give the following definition:

Definition 3. *An Ω-spectrum is a sequence of spaces $\{Z_n\}$ together with equivalences $\Omega Z_{n+1} \cong Z_n$*

Proposition 2. *There is a 1-1 correspondence between spectra and generalized cohomology theories.*

Proof. Given an Ω Spectrum $\{Z_n\}$ we can set $H^n(-) := [-, Z_n]$ and define the suspension isomorphisms by $H^{n+1}(\Sigma X) = [\Sigma X, Z_{n+1}] \cong [X, \Omega Z_{n+1}] \cong [X, Z_n] = H^n(X)$. In the other direction recall that our proof that for any cohomology theory H^n are representable using Brown's representability didn't use anywhere the dimension axiom. So to any generalized cohomology theory H^* we can assign a sequence of spaces $\{Z_n\}$ with $[-, Z_n] = H^n$. The suspension isomorphisms give us the equivalences $\Omega Z_{n+1} \cong Z_n$ (like before). □

Example 6 (K-theory)**.** - *Bott periodicity gives an equivalence $\Omega^2 BU \times \mathbb{Z} = BU \times \mathbb{Z}$ so this gives us a spectrum in the obvious way (which is moreover 2-periodic!). This spectrum is denoted KU or K and called complex K-theory. It can be computed in terms of vector bundle data on a space.*

Remark 4. *Spectra form a category with maps defined in the obvious way and we can localize this category similarly w.r.t. the homotopy equivalences similarly to what we did with spaces. The result is **not** equivalent to the category of cohomology theories! But it is very close to it! (cohomology theories are a sort of quotient by the square zero ideal consisting of "phantom maps").*