MU and FGLs

1 Spectra

There are several perspectives on what spectra are. For our purposes, the main manifestation of spectra is as the homotopical analogue of abelian groups. In particular, there is a category of spectra Sp and a diagram of functors

$$\begin{tikzcd} \mathrm{Set} \arrow[r, hook] \arrow[d, shift left, "\mathrm{Free}"] & \mathcal{S} \arrow[l, bend right, "\pi_0"'] \arrow[d, shift left, "\Sigma^\infty_+"] \\ \mathrm{Ab} \arrow[r, hook, "H"'] \arrow[u, shift left, "\mathrm{Forget}"] & \mathrm{Sp} \arrow[l, bend right, "\pi_0"'] \arrow[u, shift left, "\Omega^\infty"] \end{tikzcd}$$

Just like Ab, the ∞-category Sp admits a canonical symmetric monoidal structure $\otimes$. This makes the functor

$$\mathbb{S}\left\langle - \right\rangle : (\mathcal{S}, \times, \mathrm{pt}) \to (\mathrm{Sp}, \otimes, \mathbb{S})$$

symmetric monoidal. For a space X and a spectrum E, we have E-(co)homology of X given by:

$$E_*\left(X\right) = \pi_*\left(E \otimes \mathbb{S}\left\langle X \right\rangle\right), \quad E^* = \pi_{-*}\underline{\mathrm{hom}}\left(\mathbb{S}\left\langle X \right\rangle, E\right)$$

In particular

$$\pi_* E = E_*\left(\mathrm{pt}\right) = E_* = E^{-*} = E^*\left(\mathrm{pt}\right)$$

The homological vs. cohomological grading is really annoying, but we will not try to change the world. The construction $H\left(-\right)$ can be extended to graded abelian groups. For any graded abelian group A_* we can define

$$HA_* \simeq \bigoplus_{n \in \mathbb{Z}} H\left(A_n\right)\left[n\right].$$

2 AHSS

Let E be a spectrum. How can we understand E? first of all we have "the coefficients" E_*, which is the value of E on a point. In contrast with ordinary homology (HA for an abelian group A), this does not determine E completely. i.e. we do E need not be equal to HE_*. Just like not every space X is equivalent to $\prod K\left(\pi_n X, n\right)$.

When we try to evaluate $E^*(X)$ on a general space X (say CW complex) using the ES axioms we get the following. The suspension axiom says that the value of E on spheres is just shifts of the value on a point and that disjoint union goes to direct product. But gluing of cells will result in boundary maps that depend on E. In particular, for a general space X, we will usually have

$$E^*(X) \neq H^*(X; E^*)$$

But, we do have the AHSS spectral sequence

$$E^2 = H^*(X; E^*) \implies E^*(X)$$

that exhibits $E^*(X)$ as a *sub-quotient* of $H^*(X; E^*)$. In some cases, depending on X and E, this SS collapses and we do get an isomoprhism. This may happen for formal reasons. For example if X has cells only in even dimensions (e.g. $\mathbb{C}P^\infty = BU(1)$, $BU(1)^n$, $BU(n)$,...) and E has non-zero homotopy groups only in even dimensions (happens in many interesting cases). If in addition X has finitely many cells in each dimension, we also have that $H^*(X; \mathbb{Z})$ is free and finitely generated so by the UCT we get

$$H^*(X; E^*) = E^* \otimes H^*(X).$$

This means that evaluation of the cohomology theory $E^*(-)$ on the *object* X does not tell us anything interesting about E, but using the functoriality of $E^*(-)$ this will helps to obtain interesting invairants by evaluating $E^*(-)$ on certain *morphisms* related to X.

3 Multiplicative Structure

For a commutative ring R, the cohomology theory $H^*(-; R)$ takes values in graded commutative rings (functorially). A nicer way to say it is that HR has commutative ring structure in Sp. We can also consider a general spectrum E with a ring structure (up to homotopy). This we give functorially for every space X a graded commutative ring $E^*(X)$. In particular, the coefficients $E^*(\mathrm{pt})$ are a graded commutative ring and $E^*(X)$ is an E^*-algebra for every X.

One usually thinks of the underlying abelian group of the ring $H^*(X; R)$ as a direct sum

$$\bigoplus_{n=0}^{\infty} H^n(X; R)$$

But it will be better for our purposes to think of the direct product instead

$$\prod_{n=0}^{\infty} H^*(X; R).$$

This is the completion of the direct sum with respect to the ideal of elements of positive degree. If we remember the grading, the two objects contain the same information, because we can recover the homogeneous pieces $H^n(X; R)$.

Example 3.1. The space $\mathbb{C}P^\infty = BU(1)$ has one cell in every even degree, so $H^n(\mathbb{C}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$. We will think of the integral cohomology ring $H^*(\mathbb{C}P^\infty; \mathbb{Z})$ as the ring of formal power series $\mathbb{Z}[[t]]$ with $|t| = 2$. This is the completion of the ring of polynomials $\mathbb{Z}[t]$ that is usually thought of as the integral cohomology ring of $\mathbb{C}P^\infty$.

One drawback of the this approach is that when taking tensor products (say when applying Künneth) we need to complete again. So for example,

$$H^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; R) \simeq \mathbb{Z}[[x,y]]$$

which is not $\mathbb{Z}[[x]] \otimes \mathbb{Z}[[y]]$, but rather its completion. We shall not be careful about such things and will always implicitly complete everything that needs to be completed.

Remark 3.2. In fact, every space X is a filtered colimit of finite CW complexes $X = \varinjlim X_\alpha$ and thus $H^*(X; R) = \varprojlim H^*(X_\alpha; R)$. This filtration remembers (and refines) the grading topology.

If $E^*(X)$ is concentrated in even degrees, then forgetting the grading, it is just a commutative ring and we can consider the scheme $\operatorname{Spec} E^*(X)$. Since we are considering the "completed version", it is better to think of the "formal scheme" $\operatorname{Spf} E^*(X)$. For example we have the the formal affine line.

$$\hat{\mathbb{A}}^1 = \operatorname{Spf}(\mathbb{Z}[[t]]) = \text{"} \varprojlim \operatorname{Spec}(\mathbb{Z}[t]/t^n) \text{"}$$

should be thought of as the infinitesimal neighborhood of the zero point of the ordinary affine line $\mathbb{A}^1 = \operatorname{Spec} \mathbb{Z}[[t]]$.

4 The Formal Group Law – Rough Idea

To understand a general spectrum E, we would like to have some more invariants. Suppose X is a space with a "multiplication map"

$$m : X \times X \to X$$

This gives a map on cohomology

$$E^* X \to E^*(X \times X)$$

If for some reason we have a Kunneth isomorphism, then this is a map

$$\mu : E^* X \to E^*(X) \otimes E^*(X)$$

If m is unital / associative / commutative then μ is likewise co-unital / co-associative / co-commutative. This gives $E^* X$ a *co-ring* structure. In the situation that $E^*(X)$ is concentrated in even degrees, this defines a *multiplication* structure on $\operatorname{Spf} E^* X$ (reversing variance twice).

In particular we have $X = BU(1)$ with the map

$$BU(1) \times BU(1) \to BU(1)$$

that classifies tensor product of complex line bundles. For a ring spectrum E for which the AHSS collapses, we get a map

$$E^*[[t]] \to E^*[[x,y]]$$

given by

$$t \mapsto F(x,y)$$

where F is a formal powers series in x, y. The associativity and commutativity make F a formal group law. Namely, we get a "group structure" on the formal scheme $\hat{\mathbb{A}}^1 / E^*$.

5 Complex Orientations

Now, lets be a little bit more precise. Given a ring spectrum E, there is a simple criterion for the AHSS for $\mathbb{C}P^{\infty}$ to collapse. Note that if indeed

$$E^{*}\left(\mathbb{C}P^{\infty}\right) \simeq E^{*}\left[\left[t\right]\right]$$

then in particular we have a generator $t \in E^{2}\left(\mathbb{C}P^{\infty}\right)$ as an E^{0}-module. The map $S^{2} = \mathbb{C}P^{1} \hookrightarrow \mathbb{C}P^{\infty}$ induces a map

$$\tilde{E}^{2}\left(\mathbb{C}P^{\infty}\right) \to \tilde{E}^{2}\left(S^{2}\right) \simeq \tilde{E}^{0}\left(S^{0}\right) = E^{0}$$

the RHS has a canonical element corresponding to the unit $1 \in E^{0}$.

Claim: The AHSS for $\mathbb{C}P^{\infty}$ collapses if and only if $1 \in E^{0}$ can be lifted to $t \in \tilde{E}^{2}\left(\mathbb{C}P^{\infty}\right)$ and any such lift corresponds to an isomorphism

$$E^{*}\left(\mathbb{C}P^{\infty}\right) \simeq E^{*}\left[\left[t\right]\right]$$

The element t is called a *complex orientation* and E is called *complex orientable* if it admits a complex orientation.

When this happens we also have a Kunneth isomorphism for $\mathbb{C}P^{\infty} \times \mathbb{C}P^{\infty}$ and we get a formal group law over the ring E^{*}.

Upshot: To any complex oriented spectrum E, we can associate a formal group law. Different orientations produce isomorphic formal group laws. This turns out to be a powerful invariant of E.

Example 5.1. Some basic examples:

1. For ordinary homology $E = H\mathbb{Z}$ we get the ring $E^{*} = \mathbb{Z}$ and the additive formal group law $F\left(x, y\right) = x + y$.

2. For topological K-theory $E = KU$ we have $E^{*} = \mathbb{Z}\left[\beta^{\pm}\right]$ with $|\beta| = 2$ (the Bott element) and the multiplicative formal group law

$$F\left(x, y\right) = x + y + \beta^{-1}xy$$

3. (elliptic ?)

6 Complex Cobordism

The notion of complex orientation is somewhat awkwardly defined, but it turns out that there is a cleaner equivalent definition. First, note that complex orientations are functorial. Namely, if we have a homomorphism of commutative ring spectra $E_{1} \to E_{2}$ we have a natural map from the set of complex orientations on E_{1} to the set of complex orientations of E_{2}.

Theorem 6.1. *The functor from commutative ring spectra to sets that gives for every commutative ring spectrum E the set of complex orientations on E is (co)representable by a spectrum called MU.*

This means that there exists a universal complex oriented spectrum MU such that for every commutative ring spectrum E, to give a complex orientation on E is the same as giving a homomorphism of ring spectra $MU \to E$. In particular, E is complex orientable if and only if it admits a homomorphism of commutative ring spectra from MU. Equivalently, one can say that MU with its canonical complex orientation is initial among *complex oriented* commutative ring spectra.

We now explain a little about how this MU is constructed and how it is related to complex orientations. First, note that given a ring spectrum E, we can think of a complex orientation on E as a map

$$\eta : \mathbb{S} \to E$$

with a certain property. Namely, that it extends to a map

$$\Sigma^{-2}\mathbb{S}\left\langle \mathbb{C}P^{\infty}\right\rangle \to E.$$

There is a sequence of spectra

$$MU(0) \to MU(1) \to MU(2) \to \ldots,$$

constructed out of the spaces $BU(n)$, that allows us to define a spectrum

$$MU = \varinjlim MU(n).$$

called the *complex cobordism spectrum.* It has the following properties:

1. $MU(0) = \mathbb{S}\left\langle BU(0)\right\rangle = \mathbb{S}$.
2. $MU(1) = \Sigma^{-2}\mathbb{S}\left\langle BU(1)\right\rangle$.
3. In general $MU(n)$ is Σ^{-2n} of some "twisted version" of the suspension spectrum of $BU(n)$ using the universal bundle.
4. There are canonical maps

   $$MU(n) \times MU(m) \to MU(n+m)$$

   that turn MU into a ring spectrum (they are induced by the maps $BU(n) \times BU(m) \to BU(n+m)$ that classify direct sums of vector bundles).
5. The unit of MU as a ring spectrum is the inclusion $\mathbb{S} \simeq MU(0) \to MU$.

The essential point is that the ring MU has as a unit map $\eta : \mathbb{S} \to MU$ that by definition extends to $\Sigma^{-2}\mathbb{S}[BU(1)] \to MU$. Thus, MU is canonically complex oriented. Moreover, it is in a sense "generated freely" by the by this data as a commutative ring and so is initial.

7 Quillen's Theorem

Since MU is canonically complex oriented, there is a formal group law $F_E(x, y)$ that corresponds to it. Turns out this is the *universal* one. Namely, the formal group law is classified by a map $\varphi : L \to MU_*$.

Theorem 7.1. *(Quillen) the map $\varphi : L \to MU_*$ is an isomorphism.*

Both objects L and MU have a universal property. The ring L caries the universal formal group law and the ring spectrum MU carries the universal complex orientation. Since a complex orientation on E induces a formal group law on E^*, Quillen's theorem might seem like some abstract nonsense argument, but this is really *not* the case. Not every formal group law comes from a complex oriented spectrum so this is really a miracle. The only proofs known are computation in nature.