Caesarea Workshop Preparation 2019 - Landweber Exact Functor Theorem

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A good reference for this talk is [Lur10, Lectures 15, 16]. This talk is very similar to the talk given by Dylan Wilson at [Wil18], which has typed notes here `https://drive.google.com/file/d/15ZbcP59IgoVd3vroq1Mp2ZNgvw9wm30v/view` and a video here `https://math.colorado.edu/chtjourney/video.php?fn=Dylan_BP_5_16-360p`.

In the last talk, Lior told us about the relationship between complex orientations and formal group laws. Namely, he showed us that from a complex orientation on a ring spectrum (multiplicative cohomology theory) E, we can get a formal group law over E_* denoted $F_E(x,y) \in E_*[[x,y]]$. We saw that the spectrum $M\mathrm{U}$ carries the universal complex orientation, i.e. that giving a complex orientation is equivalent to giving a ring map $M\mathrm{U} \to E$. Furthermore, $M\mathrm{U}_* = L$ is the Lazard ring, from Stephan's talk, carrying the universal formal group law, so we get a map $L = M\mathrm{U}_* \to E_*$, classifying the formal group law over E_*.

One may wonder if can we reverse this construction. That is, can we start with a (graded) ring R and formal group law over it $F(x,y) \in R[[x,y]]$, and build a complex oriented spectrum $E_{R,F}$ from it (with coefficients $(E_{R,F})_* = R$ and formal group law $F_{E_{R,F}} = F$)? Landweber exact functor theorem describes a situation in which this is possible.

Here's a suggestion for constructing such a spectrum. We will describe it by its homology theory (it is possible to do it cohomologically). As Shaul told us, Brown representability theorem tells us that this gives a spectrum. So let's say we have a formal group law F over R. This is equivalently a map $M\mathrm{U}_* = L \to R$. We define a functor $E_{R,F} : \mathcal{S}_* \to \mathrm{GrAb}$ by:

$$(E_{R,F})_*(X) = M\mathrm{U}_*(X) \otimes_L R$$

We also have natural isomorphisms $M\mathrm{U}_n(X) \xrightarrow{\sim} M\mathrm{U}_{n+1}(\Sigma X)$, thus by tensoring with R we get $(E_{R,F})_n(X) \xrightarrow{\sim} (E_{R,F})_{n+1}(\Sigma X)$. This is a functor equipped with the suspension isomorphisms, but it is not clear that it is a homology theory. Recall Eilenberg-Steenrod axioms for homology from Shaul's talk:

1. *Additivity* - The canonical map $\bigoplus E_*(X_i) \to E_*(\bigvee_i X_i)$ is an isomorphism.

2. *Exactness* - For any inclusion of a subcomplex $A \subset X$, the associated long sequence $\cdots \to E_{n+1}(X/A) \to E_n(A) \to E_n(X) \to E_n(X/A) \to E_{n-1}(A) \to \cdots$ is exact.

Tensoring with R preserves isomorphisms, thus $E_{R,F}$ is additive. The only thing that might go wrong is exactness, that is, tensoring the LES with R might not give a long exact sequence.

One way to get exactness is that R is flat over L (which precisely means that tensoring with R over L, sends an exact sequences of L modules to an exact sequence). However, recall that Lazard's theorem says that $L \cong \mathbb{Z}[x_1, x_2, \dots]$ which is a very big ring. Therefore, it is hard to be flat over it. It turns out that this condition is indeed stronger than necessary, and can be relaxed. Moreover, the relaxed condition is very easy to check, which is also a big plus.

First we consider a simplified situation of what really happens. Imagine that we had a ring map $L' \to L$, s.t. all of the modules in the exactness axiom were base-changed from L', thus our modules are not arbitrary L-modules. Then, we can consider R over L' (instead of L), and check the flatness only over L'. If L' was a "smaller" ring than L, it might be considerably easier to be flat over it. Our situation is not too far from this.

I do not wish to get into it, but I will say quickly for the algebro-geometrically inclined people. Instead of a map $\operatorname{Spec} L \to \operatorname{Spec} L'$ what we a have is map $\operatorname{Spec} L \to \mathcal{M}_{\mathrm{fg}}$, where the latter is the moduli stack of formal groups, thus we just need to check flatness over it.

Let E be a ring spectrum, and X a space. First we consider the cohomology $E^*(X) = \pi_{-*} \hom(\Sigma^\infty X, E)$. This is a priori only an E^*-module. However, there are cohomology operations, which are given by maps from E to itself, that is $E^*E = \pi_{-*} \hom(E, E)$. This gives an *action* of E^*E on $E^*(X)$ over E^* by post-composition, $E^*(X) \otimes_{E^*} E^*E \to E^*(X)$. This makes the E^*-module $E^*(X)$ a module over the algebra E^*E. Now, consider the homology $E_*(X) = \pi_*(\Sigma^\infty X \otimes E)$, which is of course a module over E_*. Dually, (under some conditions) it has *coaction* by $E_*E = \pi_*(E \otimes E)$ over E_*, $\Delta : E_*(X) \to E_*(X) \otimes_{E_*} E_*E$. Here the map is obtained by tensoring the unit map $\mathbb{S} \to E$ with E. This makes the E_*-module $E_*(X)$ a comodule over the coalgebra E_*E.

In our situation, this means that $M\mathrm{U}_*(X)$ is not merely an $M\mathrm{U}_* = L$-module, but also an $M\mathrm{U}_*M\mathrm{U} = \Gamma$-comodule. Thus we only need to know that $- \otimes_L R : \mathrm{Comod}_{(L,\Gamma)} \xrightarrow{\text{forgetful}} \mathrm{Mod}_L \to \mathrm{Mod}_R$ is exact, that is, sends exact sequences of modules coming from such comodules to exact sequences. Landweber gave a very checkable condition, equivalent to the exactness of that functor, in terms of the formal group law F/R classified by $L \to R$.

Recall that for a formal group law F over R, we defined the m-series, $[m]_F(x) = \underbrace{X +_F \cdots +_F X}_{m \text{ times}} \in R[[x]]$, which is the multiplication by m.

Definition 1. Let p be a prime, we define $v_n \in R$ as the coefficient of x^{p^n} in the p-series $[p]_F(x)$ (this depends on R, F, p which are omitted from the notation).

We note that $v_0 = p$ is the coefficient of $x = x^{p^0}$. As a reminder, although not very relevant now, the formal group law (over a field of characteristic p) was of height n if $v_i = 0$ for $i < n$ and $v_n \neq 0$. We should note that this v_n is *not* an invariant of F (and in fact not quite the right definition), it might change under an isomorphism, but it is invariant modulo $(p = v_0, v_1, \ldots, v_{n-1})$.

Now, for the formal group law F over R, for every prime p we have the sequence $p = v_0, v_1, v_2 \ldots$, and we can finally define Landweber flatness:

Definition 2. The formal group law F over R is called *Landweber flat*, if and only if, for every prime p and n, the map $R/(v_0, v_1, \ldots, v_{n-1}) \xrightarrow{\times v_n} R/(v_0, v_1, \ldots, v_{n-1})$ is injective, i.e. not a zero-divisor. In other words, the sequence $p = v_0, v_1, v_2 \ldots$ is regular.

We note that 0 is *not* a zero-divisor in the 0 ring. Moreover, if v_n is invertible in $R/(v_0, v_1, \ldots, v_{n-1})$ it is certainly not a zero-divisor, and we get that $R/(v_0, v_1, \ldots, v_{n-1}, v_n) = 0$, so the condition holds for v_k for $k \geq n+1$.

Armed with this definition we can finally state the main theorem:

Theorem 3 (Landweber Exact Functor Theorem (LEFT))**.** *If F over R is Landweber flat, then $(E_{R,F})_*(X) = M\mathrm{U}_*(X) \otimes_L R$ is a homology theory. (In fact, Landweber flatness is equivalent to the functor $- \otimes_L R$ from comodules being exact.)*

We now give examples of a few Landweber flat and non-flat formal group laws.

First, a class of examples, let R be some ring, and consider the additive formal group law over it, that is $F_a(x, y) = x + y$. The p-series is then $[p]_{F_a}(x) = px$, thus $v_0 = p$ and $v_n = 0$ for $n > 0$. (We can immediately conclude that it is Landweber flat only over $\mathbb{Q}$-algebras, but let's see a few examples.)

Example 4. For $R = \mathbb{Q}$, F_a is Landweber flat. Indeed for every p, $v_0 = p$ is invertible in $\mathbb{Q}$ and in particular not a zero-divisor, and $\mathbb{Q}/p = 0$, so as we said the condition holds for v_n for $n \geq 1$. This example gives $M\mathrm{U}_*(X) \otimes_L \mathbb{Q} = H_*(X; \mathbb{Q})$ ordinary homology over $\mathbb{Q}$!

Example 5. For $R = \mathbb{F}_q$, F_a is *not* Landweber flat. Here primes $p \neq q$ do work, but for the prime q, $v_0 = q = 0$ in $\mathbb{F}_q$, which is a zero-divisor.

Example 6. For $R = \mathbb{Z}$, F_a is *not* Landweber flat. For every prime p, $v_0 = p$ is not zero-divisor in $\mathbb{Z}$, but $v_1 = 0$ is a zero-divisor in $\mathbb{F}_p$.

Example 7. We now reconstruct K-theory! Let $R = \mathbb{Z}\left[\beta^{\pm 1}\right]$, and consider the multiplicative formal group law over it (coming from K-theory), $F_m(x,y) = x + y + \beta xy = \beta^{-1}\left((1+\beta x)(1+\beta y) - 1\right)$. Then the p-series is $[p]_{F_m}(x) = \beta^{-1}\left((1+\beta x)^p - 1\right) = px + \underbrace{\cdots}_{\text{don't care}} + \beta^{p-1}x^p$, thus $v_0 = p, v_1 = \beta^{p-1}$ and higher v_n's vanish. Indeed, $v_0 = p$ is not a zero-divisor in $R = \mathbb{Z}\left[\beta^{\pm 1}\right]$. Furthermore, $v_1 = \beta^{p-1}$ is invertible, thus not zero-divisor, in $R/p = \mathbb{F}_p\left[\beta^{\pm 1}\right]$. And, as we said, since it is invertible, the condition holds for v_n for $n \geq 2$. It turns out that indeed this recovers K-theory as $K_*(X) = M\mathrm{U}_*(X) \otimes_L \mathbb{Z}\left[\beta^{\pm 1}\right]$.

Proof sketch of LEFT. Recall that we wanted to show that if F over R is Landweber flat, then $- \otimes_L R : \mathrm{Comod}_{(L,\Gamma)} \xrightarrow{\text{forgetful}} \mathrm{Mod}_L \to \mathrm{Mod}_R$ is exact. ☐

1. Flatness over L is equivalent to $\mathrm{Tor}_L(R, M) = 0$ for every $M \in \mathrm{Mod}_L$.
2. We can filter M, s.t. the associated graded is $\mathrm{gr}_* M = \bigoplus L/\mathfrak{p}_i$, for prime ideals . So it is enough to check $\mathrm{Tor}_L(R, L/\mathfrak{p}) = 0$ for every prime ideal.
3. If we consider only modules which come from Γ-comodules, we can arrange the filtration s.t. each $L/\mathfrak{p}$ is invariant under the coaction, that is $\Delta(\mathfrak{p}) \subseteq \mathfrak{p} \otimes_L \Gamma$.
4. Landweber proved the *Invariant Prime Ideal theorem*, which says that $(p = v_0, v_1, \ldots, v_n)$ are the only invariant prime ideals.
5. Applying this, we can show that the vanishing of $\mathrm{Tor}_L(R, L/(v_0, v_1, \ldots, v_n))$ for all n is equivalent to the series being regular in R.

References

[Lur10] J. Lurie. "Chromatic homotopy theory". In: *252x course notes,* `http://www.math.harvard.edu/~lurie/252x.html` (2010).

[Wil18] Dylan Wilson. "The Landweber exact functor theorem". In: *Chromatic homotopy theory: Journey to the Frontier,* `https://sites.google.com/view/chtjourney` (2018).