Integral Models for Spaces via the Higher Frobenius

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1 Introduction and Outline

Let me being by reminding two of the main theorems we have seen so far.

Theorem 1 (Sullivan). *The functor $\mathbb{Q}^{(-)}\colon \left(\mathcal{S}_{\mathbb{Q}}^{\omega,\geq 2}\right)^{\mathrm{op}} \to \mathrm{CAlg}_{\mathbb{Q}}$ is fully faithful. In particular, $X_{\mathbb{Q}} = \mathrm{Map}_{\mathbb{Q}}\left(\mathbb{Q}^X, \mathbb{Q}\right)$ for $X \in \mathcal{S}^{\omega,\geq 2}$.*

Theorem 2 (Mandell). *The functor $\overline{\mathbb{F}}_p^{(-)}\colon \left(\mathcal{S}_p^{\omega,\geq 2}\right)^{\mathrm{op}} \to \mathrm{CAlg}_{\overline{\mathbb{F}}_p}$ is fully faithful. In particular, $X_p = \mathrm{Map}_{\overline{\mathbb{F}}_p}\left(\overline{\mathbb{F}}_p^X, \overline{\mathbb{F}}_p\right)$ for $X \in \mathcal{S}^{\omega,\geq 2}$.*

The goal of this talk is to combine these results together with the Frobenius explained in the previous talk, to present Allen Yuan's results for an integral model for (finite simply-connected) spaces. Our starting point is the arithmetic square, which essentially says that integral data is obtained by p-complete information, rational information, and *gluing information:*

Theorem 3 (Sullivan). *Let $X \in \mathcal{S}^{\omega,\geq 2}$, then there is a pullback square:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X_p \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X_p\right)_{\mathbb{Q}} \end{tikzcd}$$

The problem is that we don't have a model for the bottom-right corner and the maps. If we rationalize the p-adic model we get 0, as $\overline{\mathbb{F}}_p \otimes \mathbb{Q} = 0$, so something has to be modified. Allen's solution involves both lifting to mixed characteristic (i.e. $\overline{\mathbb{Z}}_p = W\overline{\mathbb{F}}_p$ or rather a lift to spectra $\overline{\mathbb{S}}_p$), and taking the Frobenius fixed points (and remember that we fixed them!). We will start with a very rough outline of the paper, and then fill in (some of) the details.

Theorem (A)**.** *There is a category* $\mathrm{CAlg}_p^{\mathrm{perf}} \subseteq \mathrm{CAlg}_p$ *of* p-perfect algebras*, admitting an action of* S^1 *such that the monodromy of the action is the Frobenius* $\varphi\colon R \xrightarrow{\sim} R$*.*

Example 4. $\mathbb{S}_p \in \mathrm{CAlg}_p^{\mathrm{perf}}$ is a p-perfect algebra.

As in ordinary algebra, we can take the Frobenius fixed points. However, in this case the trivialization is *extra information*, so that there is a category $\mathrm{CAlg}_p^{\varphi=1}$ of *p-Frobenius fixed algebras*. We shall see that the p-complete sphere $\mathbb{S}_p$ is canonically p-Frobenius fixed, i.e. that there is a *canonical* trivialization of its Frobenius, giving rise to an object $\mathbb{S}_{p,\varphi=1} \in \mathrm{CAlg}_p^{\varphi=1}$.

Theorem (B)**.** *The functor* $\mathbb{S}_{p,\varphi=1}^{(-)}\colon \left(\mathcal{S}_p^{\omega,\geq 2}\right)^{\mathrm{op}} \to \mathrm{CAlg}_p^{\varphi=1}$ *is fully faithful.*

The last step is to globalize, and as we shall see it is enough to trivialize the p-Frobenius in each prime separately. Namely, let $\mathrm{CAlg}^{\mathrm{perf}} \subseteq \mathrm{CAlg}$ be the category of *perfect algebras*, i.e. algebras that are p-perfect after p-completion for all p. Let $\mathrm{CAlg}^{\varphi=1}$ be the category of *Frobenius fixed* algebras, i.e. perfect algebras together with a trivialization of the p-Frobenius for each p. Since each $\mathbb{S}_p$ is canonically p-Frobenius fixed, we get that $\mathbb{S}$ is canonically Frobenius fixed, giving rise to $\mathbb{S}_{\varphi=1} \in \mathrm{CAlg}^{\varphi=1}$.

Theorem (C)**.** *The functor* $\mathbb{S}_{\varphi=1}^{(-)}\colon \left(\mathcal{S}^{\omega,\geq 2}\right)^{\mathrm{op}} \to \mathrm{CAlg}^{\varphi=1}$ *is fully faithful.*

2 Frobenius - Theorem A

Recall that in ordinary algebra, for any commutative ring R, we have the Frobenius $\varphi_R\colon R \to R/p$ and the canonical map $\mathrm{can}_R\colon R \to R/p$. The map can_R is an isomorphism iff R is an $\mathbb{F}_p$-algebra, in which case we can consider the Frobenius morphism as an automorphism. We then restrict to the perfect algebras (those where Frobenius is invertible), such as $\mathbb{F}_p$ and $\overline{\mathbb{F}}_p$, $\mathrm{CAlg}_{\mathbb{F}_p}^{\mathrm{perf}} \subseteq \mathrm{CAlg}_{\mathbb{F}_p}$. Furthermore, on some of those algebras, the Frobenius is trivial, e.g. $\mathbb{F}_p$, and we let $\mathrm{CAlg}_{\mathbb{F}_p}^{\varphi=1} \subseteq \mathrm{CAlg}_{\mathbb{F}_p}^{\mathrm{perf}}$ be their subcategory. We thus get an adjunction $\mathrm{CAlg}_{\mathbb{F}_p}^{\varphi=1} \overset{(-)^{\mathbb{Z}}}{\leftrightarrows} \mathrm{CAlg}_{\mathbb{F}_p}^{\mathrm{perf}}$, where the right adjoint is given by the Frobenius fixed points $R^{\mathbb{Z}} = \{x \in R \mid \varphi(x) = x\}$.

Lior explained how to extend this to higher algebra. We have seen that for any commutative algebra there is a ring map $\mathrm{can}_R\colon R \to R^{tC_p}$ as well as $\varphi_R\colon R \to R^{tC_p}$. We then defined the full subcategory of p-perfect algebras $\mathrm{CAlg}_p^{\mathrm{perf}} \subseteq \mathrm{CAlg}_p$ consisting of those p-complete algebras that admit an invertible Frobenius.

Example 5. $\mathbb{S}_p$ and $\overline{\mathbb{S}}_p$ are in $\mathrm{CAlg}_p^{\mathrm{perf}}$, i.e. admit an invertible Frobenius.

As we have seen in Lior's talk, one of Allen's main results is:

Theorem (A)**.** *S^1 acts on $\mathrm{CAlg}_p^{\mathrm{perf}}$, i.e. there is a diagram $BS^1 \to \mathrm{Cat}$, such that the monodromy of the action is the Frobenius $\varphi\colon R \xrightarrow{\sim} R$.*

Definition 6. $\mathrm{CAlg}_p^{\varphi=1} = \left(\mathrm{CAlg}_p^{\mathrm{perf}}\right)^{hS^1}$, the category of *$p$-Frobenius fixed algebras.*

Corollary 7. *There is an adjunction $\mathrm{CAlg}_p^{\varphi=1} \overset{(-)^{h\mathbb{Z}}}{\leftrightarrows} \mathrm{CAlg}_p^{\mathrm{perf}}$ where the (underlying algebra of) $R^{h\mathbb{Z}}$ is the fixed points by the Frobenius.*

The explanation of the definition and this corollary is, I think, the most confusing part of the lecture. If you lose me you can just take this as a blackbox, but I will now try to explain how this works by starting with a simpler case.

Consider the case of a finite group G acting on a category $\mathcal{C}$, that is we have a diagram $BG \to \mathrm{Cat}$ where the point is sent to $\mathcal{C}$. In particular, for any object $X \in \mathcal{C}$ and element $g \in G$ we have another object $gX \in \mathcal{C}$. Therefore, an object of the fixed point category $\mathcal{C}^{hG} = \lim_{BG}(BG \to \mathrm{Cat})$ is an object $X \in \mathcal{C}$ together with identifications $gX \simeq X$ for every $g \in G$ (note that it doesn't make sense to ask that $gX = X$ in a category). There is a forgetful map $\mathcal{C}^{hG} \to \mathcal{C}$, sending $\left(X, (X \simeq gX)_g\right)$ to X. Somewhat surprisingly, this functor has a right adjoint which I want to describe. Given an object $X \in \mathcal{C}$, the product of the orbit $\prod_{h\in G} hX$. Remember that for any $g \in G$ we get another object, and in this case $g\prod_G hX = \prod_G ghX$. As g acts by permuting the copies, there is a canonical identification between $\prod hX \simeq \prod ghX$, assembling together into an object of $\mathcal{C}^{hG}$. To generalize, note that $\prod_{h\in G} hX = \lim_G X$. Indeed, this can be made into an adjoint $\mathcal{C} \xrightarrow{\lim_G(-)} \mathcal{C}^{hG}$.

In the case $G = S^1$, everything works exactly the same. To think about this informally, picture $S^1 = * \rightrightarrows *$, so that the action is determined by a functor $F\colon \mathcal{C} \to \mathcal{C}$ (in our case $(-)^{tC_p}$) and two natural isomorphisms $\psi\colon \mathrm{id} \Rightarrow F, \varphi\colon \mathrm{id} \Rightarrow F$ (in our case can and φ). A fixed point in $\mathcal{C}^{hS^1}$ should then have an identification $X \simeq FX$, identified with ψ and φ. All in all then, this is the data of the identification of $\psi \simeq \varphi$ (in fact there is more structure, by the compositions, e.g. $F^{\circ 2}$ and $\psi_{FX}, F\psi_X, \varphi_{FX}, F\varphi_X$ and so on). As above, given $X \in \mathcal{C}$, the limit $\lim_{S^1} X$ thus admits the structure of an object in $\mathcal{C}^{hS^1}$. To sum up, there is a forgetful map $\mathcal{C}^{hS^1} \to \mathcal{C}$ and it has a right adjoint given by $\mathcal{C} \xrightarrow{\lim_{S^1}(-)=(-)^{h\mathbb{Z}}} \mathcal{C}^{hS^1}$.

3 Algebraic Extensions and "Frobenius Descent"

Remark 8. What follows is *not* the ordinary Galois descent of modules you might be familiar with, although it is related.

In ordinary algebra, given $R \in \mathrm{CAlg}_{\mathbb{F}_p}^{\mathrm{perf}}$ we can consider $R \otimes \overline{\mathbb{F}}_p \in \mathrm{CAlg}_{\overline{\mathbb{F}}_p}^{\mathrm{perf}}$, a perfect

$\overline{\mathbb{F}}_p$-algebra, yielding a pair of adjunctions:

$$\mathrm{CAlg}^{\varphi=1}_{\mathbb{F}_p} \overset{(-)^{\mathbb{Z}}}{\leftrightarrows} \mathrm{CAlg}^{\mathrm{perf}}_{\mathbb{F}_p} \underset{-\otimes\overline{\mathbb{F}}_p}{\leftrightarrows} \mathrm{CAlg}^{\mathrm{perf}}_{\overline{\mathbb{F}}_p}$$

Theorem 9. *The composition* $\mathrm{CAlg}^{\varphi=1}_{\mathbb{F}_p} \to \mathrm{CAlg}^{\mathrm{perf}}_{\overline{\mathbb{F}}_p}$ *is fully faithful.*

Proof. This is equivalent to showing that the unit map of the adjunction is an isomorphism. Let $R \in \mathrm{CAlg}^{\varphi=1}_{\mathbb{F}_p}$, the Frobenius of $R \otimes \overline{\mathbb{F}}_p$ is $\varphi_{R\otimes\overline{\mathbb{F}}_p} = \varphi_R \otimes \varphi_{\overline{\mathbb{F}}_p} = \mathrm{id} \otimes \varphi_{\overline{\mathbb{F}}_p}$, so that the fixed points are indeed $\left(R \otimes \overline{\mathbb{F}}_p\right)^{\mathbb{Z}} = R$. $\square$

As explained by Lior, there is a lift of $\mathbb{F}_p \to \overline{\mathbb{F}}_p$ denoted $\mathbb{S}_p \to \overline{\mathbb{S}}_p$, where $\overline{\mathbb{S}}_p$ is called the *spherical Witt vectors.* It satisfies $\pi_0\overline{\mathbb{S}}_p/p = \overline{\mathbb{F}}_p$, and has a Frobenius (i.e. p-perfect $\overline{\mathbb{S}}_p \in \mathrm{CAlg}^{\mathrm{perf}}_p$) whose reduction is the Frobenius on $\overline{\mathbb{F}}_p$ (and in a way it is determined by these properties). As in ordinary algebra, we get an adjunction:

$$\mathrm{CAlg}^{\varphi=1}_{p} \overset{(-)^{h\mathbb{Z}}}{\leftrightarrows} \mathrm{CAlg}^{\mathrm{perf}}_{p} \underset{-\otimes\overline{\mathbb{S}}_p}{\leftrightarrows} \mathrm{CAlg}^{\mathrm{perf}}_{\overline{\mathbb{S}}_p}$$

Definition 10. We denote $\mathbb{S}_{p,\varphi=1} := \overline{\mathbb{S}}_p^{h\mathbb{Z}}$, i.e. the fixed points by the Frobenius.

Proposition 11. *Forgetting the trivialization of the Frobenius on* $\mathbb{S}_{p,\varphi=1}$ *gives* $\mathbb{S}_p$.

Proof. We know that $\overline{\mathbb{S}}_p^{h\mathbb{Z}} = \mathrm{eq}\left(\overline{\mathbb{S}}_p \underset{\varphi}{\overset{1}{\rightrightarrows}} \overline{\mathbb{S}}_p\right)$. The composition of the unit map $\mathbb{S}_p \to \overline{\mathbb{S}}_p$ with 1 and with φ is still the unit map (since both are trivial on $\mathbb{S}_p$), so it maps to the equalizer. Thus we need to show that $\mathbb{S}_p \xrightarrow{\sim} \mathrm{eq}\left(\overline{\mathbb{S}}_p \underset{\varphi}{\overset{1}{\rightrightarrows}} \overline{\mathbb{S}}_p\right)$, which follows from the fact $\pi_*\overline{\mathbb{S}}_p = \pi_*\mathbb{S}_p \otimes_{\mathbb{Z}_p} \overline{\mathbb{Z}}_p$ and the fact that Frobenius acts only on the $\overline{\mathbb{Z}}_p$ part. $\square$

Theorem 12. *The composition* $- \otimes \overline{\mathbb{S}}_p : \mathrm{CAlg}^{\varphi=1}_p \to \mathrm{CAlg}^{\mathrm{perf}}_{\overline{\mathbb{S}}_p}$ *is fully faithful.*

Proof. Again it suffices to show that the unit is an isomorphism, so we show that for any $R_{\varphi=1} \in \mathrm{CAlg}^{\varphi=1}_p$ we have $R_{\varphi=1} \xrightarrow{\sim} \left(R \otimes \overline{\mathbb{S}}_p\right)^{h\mathbb{Z}}$. Since the Frobenius is trivial on R, it easy to show that $\left(R \otimes \overline{\mathbb{S}}_p\right)^{h\mathbb{Z}} = R \otimes \overline{\mathbb{S}}_p^{h\mathbb{Z}}$, so the result follows from the previous proposition. $\square$

4 A p-adic Model

Proposition 13. *The functor* $\overline{\mathbb{S}}_p^{(-)} \colon \left(\mathcal{S}_p^{\omega,\geq 2}\right)^{\mathrm{op}} \to \mathrm{CAlg}_{\overline{\mathbb{S}}_p}^{\mathrm{perf}}$ *is fully faithful.*

Proof. The claim will follow from Mandell's theorem if we show that the following map is an equivalence

$$\mathrm{Map}_{\overline{\mathbb{S}}_p}\left(\overline{\mathbb{S}}_p^X, \overline{\mathbb{S}}_p^Y\right) \xrightarrow{\sim} \mathrm{Map}_{\overline{\mathbb{F}}_p}\left(\overline{\mathbb{F}}_p^X, \overline{\mathbb{F}}_p^Y\right).$$

As both sides take colimits in Y to limits, it suffices to prove this for $Y = *$. By adjunction, the RHS is $\mathrm{Map}_{\overline{\mathbb{S}}_p}\left(\overline{\mathbb{S}}_p^X, \overline{\mathbb{F}}_p\right)$. Therefore we need to show that the result does not change if we switch from $\overline{\mathbb{F}}_p$ to $\overline{\mathbb{S}}_p$. I will not explain this, but this is a similar rigidity phenomenon to Hensel's lemma. □

5 Another p-adic Model - Theorem B

Definition 14. We define the functor $\mathbb{S}_{p,\varphi=1}^{(-)} \colon \left(\mathcal{S}_p^{\omega,\geq 2}\right)^{\mathrm{op}} \to \mathrm{CAlg}_p^{\varphi=1}$ as the composition X to $\mathbb{S}_{p,\varphi=1}^X := \left(\overline{\mathbb{S}}_p^X\right)^{h\mathbb{Z}}$.

Theorem (B)**.** *The functor* $\mathbb{S}_{p,\varphi=1}^{(-)}$ *is fully faithful.*

Proof. Note that the Frobenius on $\mathbb{S}_p^X$ is trivial because of the naturality of the Frobenius, so that $\overline{\mathbb{S}}_p^X$ is in the essential image of the fully faithful functor $- \otimes \overline{\mathbb{S}}_p : \mathrm{CAlg}_p^{\varphi=1} \to \mathrm{CAlg}_{\overline{\mathbb{S}}_p}^{\mathrm{perf}}$. The result then follows because

$$\mathrm{Map}_{\mathbb{S}_{p,\varphi=1}}\left(\mathbb{S}_{p,\varphi=1}^X, \mathbb{S}_{p,\varphi=1}^Y\right) = \mathrm{Map}_{\overline{\mathbb{S}}_p}\left(\overline{\mathbb{S}}_p^X, \overline{\mathbb{S}}_p^Y\right) = \mathrm{Map}\left(Y, X\right).$$

□

6 Integral Model - Theorem C

We define $\mathrm{CAlg}^{\mathrm{perf}} \subseteq \mathrm{CAlg}$ as the category of *perfect algebras*, i.e. algebras that are p-perfect after p-completion for all p. In addition, we define the category $\mathrm{CAlg}^{\varphi=1}$ of *Frobenius fixed* algebras as the pullback

$$\begin{tikzcd}
\mathrm{CAlg}^{\varphi=1} \arrow[r] \arrow[d] & \prod_p \mathrm{CAlg}_p^{\varphi=1} \arrow[d] \\
\mathrm{CAlg}^{\mathrm{perf}} \arrow[r] & \prod_p \mathrm{CAlg}_p^{\mathrm{perf}}
\end{tikzcd}$$

i.e., perfect algebras with a trivialization of the p-Frobenius for each p separately. We obtain the object $\mathbb{S}_{\varphi=1} \in \mathrm{CAlg}^{\varphi=1}$ from $\mathbb{S}$ and $\mathbb{S}_{p,\varphi=1}$. The functors $\mathbb{S}_{p,\varphi=1}^{(-)}$ and $\mathbb{S}^{(-)}$ then assemble to a functor $\mathbb{S}_{\varphi=1}^{(-)}\colon \left(\mathcal{S}^{\omega,\geq 2}\right)^{\mathrm{op}} \to \mathrm{CAlg}^{\varphi=1}$.

Theorem (C)**.** *The functor* $\mathbb{S}_{\varphi=1}^{(-)}\colon \left(\mathcal{S}^{\omega,\geq 2}\right)^{\mathrm{op}} \to \mathrm{CAlg}^{\varphi=1}$ *is fully faithful.*

Proof. We check that the unit is an isomorphism, i.e. that $X \xrightarrow{\sim} \mathrm{Map}_{\mathbb{S}_{\varphi=1}}\left(\mathbb{S}_{\varphi=1}^X, \mathbb{S}_{\varphi=1}\right)$. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Map}_{\mathbb{S}_{\varphi=1}}\left(\mathbb{S}_{\varphi=1}^X, \mathbb{S}_{\varphi=1}\right) \arrow[r] \arrow[d] & \prod_p \mathrm{Map}_{\mathbb{S}_{p,\varphi=1}}\left(\mathbb{S}_{p,\varphi=1}^X, \mathbb{S}_{p,\varphi=1}\right) \arrow[d] \\
\mathrm{Map}_{\mathbb{S}}\left(\mathbb{S}^X, \mathbb{S}\right) \arrow[r] \arrow[d] & \prod_p \mathrm{Map}_{\mathbb{S}_p}\left(\mathbb{S}_p^X, \mathbb{S}_p\right) \arrow[d] \\
\mathrm{Map}_{\mathbb{Q}}\left(\mathbb{Q}^X, \mathbb{Q}\right) \arrow[r] & \mathrm{Map}_{\mathbb{S}}\left(\mathbb{S}^X, \left(\prod_p \mathbb{S}_p\right)_{\mathbb{Q}}\right)
\end{tikzcd}$$

The upper square is a pullback square by definition of the category $\mathrm{CAlg}^{\varphi=1}$. The bottom square is a pullback square, obtained by adjunctions and applying $\mathrm{Map}_{\mathbb{S}}\left(\mathbb{S}^X, -\right)$ to the arithmetic square $\mathbb{S} = \prod_p \mathbb{S}_p \times_{\left(\prod_p \mathbb{S}_p\right)_{\mathbb{Q}}} \mathbb{Q}$. Therefore the big square is a pullback square.

By the rational model, the bottom left is a $X_{\mathbb{Q}}$. By Allen's p-adic model, the upper right corner is $\prod_p X_p$. By more results proved by Mandell, the bottom right corner is $\left(\prod_p X_p\right)_{\mathbb{Q}}$ (and the two maps are the expected once). Therefore, by the arithmetic square of X, the top left corner is indeed X. $\square$