HHR Strategy - G Seminar

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We are going to have E_∞-rings, E_n-pages of spectral sequences and Morava E-Theory E_n. To avoid at least a little of the confusion, I call E_∞-rings simply ring spectra.

We consider θ_j and h_j^2, which live in the 2-local parts of the homotopy groups of spheres. Therefore, we can 2-localize all of our spectra, and we won't write this every time.

The main reference for this is the paper [HHR09]. Also very useful are [HHR10; Mil11]. Lastly, in 2016 there was a Talbot on this topic [Tal16], with full (typed!) notes by Eva Belmont.

1 Idea of the Proof

Recall that in Shaul's talk we saw:

Theorem (Browder)**.** *A manifold of Kervaire invariant one exists if and only if* $h_j^2 \in E_2^{2,2^{j+1}}(\mathbb{S}, H\mathbb{F}_2) = \mathrm{Ext}_{\mathcal{A}}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ *survives to the* E_∞*-page, i.e. it supports an element* $\theta_j \in \pi_{2^{j+1}-2}\mathbb{S}$*.*

The elements θ_j for $1 \leq j \leq 5$ were constructed by explicit computations. The purpose of HHR's paper is to show:

Theorem ([HHR09, Theorem 1.1])**.** *For* $j \geq 7$*,* h_j^2 *doesn't survive to the* E_∞*-page, i.e. the element* θ_j doesn't *exist.*

The bottom line of the proof is very simple. Find a spectrum with a map $\mathbb{S} \to \Omega$ such that it *detects* θ_j, i.e. if θ_j exists its images is non-zero, and show that that $\pi_{2^{j+1}-2}\Omega = 0$, which together contradict the existence of θ_j. To be more precise, the proof of theorem is as follows. First, we construct some spectrum Ω together with a map $\mathbb{S} \to \Omega$, and prove the following 3 theorems (for $j \geq 7$):

Theorem (Detection)**.** *If* θ_j *exists, its image in* $\pi_{2^{j+1}-2}\Omega$ *is non-zero.*

Theorem (Gap)**.** $\pi_{-2}\Omega = 0$ *(in fact π_{-1}, π_{-3} vanish as well).*

Theorem (Periodicity)**.** $\pi_i\Omega = \pi_{i+2^{7+1}}\Omega$ *(in fact $\Omega^i(X)$ is 2^{7+1}-periodic for all X).*

Each one of these constitutes a major part of the proof, and together the immediately imply the result above. We will devote the next lectures to proving these theorems. Today we will motivate and construct Ω (to some extent), and discuss some aspects of these theorems.

2 Motivating Ω

Say you have an element in $\pi_i\mathbb{S}$. How can you detect it, i.e. show that it is non-zero? One standard method is as above, find some map $\mathbb{S} \to A$ under which you can prove that the element is not mapped to 0.

Everything that follows is just motivation for the construction of Ω, I hope this will be at least somewhat clear to everyone, but don't worry if it isn't, as it is just motivation.

Let's take a simpler case then $\theta_j, j \geq 7$, which will shed some light on what's going on in this paper. Say we want to detect $\theta_1 \in \pi_{2^{1+1}-2}\mathbb{S} = \pi_2\mathbb{S}$ (which is indeed non-zero). This element is the square of the Hopf fibration $\eta : S^3 \to S^2$ (represented in the ASS by h_1), i.e. $\theta_j = \eta^2 : S^4 \xrightarrow{\Sigma\eta} S^3 \xrightarrow{\eta} S^2$. A way to detect it is using $K\mathrm{O}$. Indeed, $K\mathrm{O}$ is a ring and in particular has a map $\mathbb{S} \to K\mathrm{O}$, and it is classical that the image of η^2 generates $\pi_2 K\mathrm{O} \cong \mathbb{Z}/2$, and in particular it's image is non-zero, thus η^2 is non-zero (see for example [Sch07, page 22]).

We seek to generalize this, so we should understand where $K\mathrm{O}$ comes from. One can construct (we won't but we will do something similar later) a C_2-ring spectrum $K\mathrm{U}_{\mathbb{R}}$, whose underlying spectrum is $\underline{K\mathrm{U}_{\mathbb{R}}} = K\mathrm{U}$, and $K\mathrm{U}_{\mathbb{R}}^{hC_2} = K\mathrm{O}$. Since it is a C_2-ring spectrum, we get the map $\mathbb{S} \to K\mathrm{U}_{\mathbb{R}}^{hC_2} = K\mathrm{O}$ from above.

This lends itself to a generalization. Consider Morava E-theory E_n (at prime 2), the case $n = 1$ being $E_1 = K\mathrm{U}$. These are known to be ring spectra acted by the Morava Stabilizer group $\mathbb{G}_n$ via ring maps (Goerss-Hopkins-Miller-Lurie). Similarly to the above, we can choose our favorite subgroup $G < \mathbb{G}_n$, and consider the map $\mathbb{S} \to E_n^{hG}$. This can potentially detect some new elements.

Furthermore, using the proof of the Nilpotence Theorem (vanishing line), one can show that the homotopy groups of such E_n^{hG} are periodic for some power of 2, which would yield the Periodicity Theorem (at least for some power of 2).

Lastly, it was known to HHR (see [HHR10, Remark 6.10]) that π_{-2} (and π_{-3}, π_{-1}) vanish for $K\mathrm{O} = E_1^{hC_2}$ and for $E_2^{hC_4}$ (unpublished), which is a promising hint towards the Gap Theorem.

It looked to HHR as though $E_4^{hC_8}$, for $C_8 \leq \mathbb{G}_4$ could serve as Ω in their proof. However, it turned out that actually showing that $\pi_{-2} E_4^{hC_8} = 0$ was a very hard computation using the homotopy fixed points spectral sequence (see [HHR10, p. 3.3]).

Instead, they decided to look for another spectrum. Recall the map $\mathbb{S} \to K\mathrm{U}_{\mathbb{R}}$ which gave us $\mathbb{S} \to K\mathrm{U}_{\mathbb{R}}^{hC_2} = K\mathrm{O}$. In fact, this map factors as $\mathbb{S} \to M\mathrm{U}_{\mathbb{R}} \to K\mathrm{U}_{\mathbb{R}}$, where $M\mathrm{U}_{\mathbb{R}}$ is $M\mathrm{U}$ with a genuine C_2 action (which we will construct later), and the second map is the complex orientation (furthermore, $K\mathrm{U}_{\mathbb{R}}$ can be constructed from $M\mathrm{U}_{\mathbb{R}}$). This gives us a map $\mathbb{S} \to M\mathrm{U}_{\mathbb{R}}^{hC_2} \to K\mathrm{U}_{\mathbb{R}}^{hC_2} = K\mathrm{O}$, so this is enough to detect η^2. We can mimic this. Recall that E_4 is equipped with an action of C_8, and we can in fact promote it to a genuine C_8-ring. Consider the map $\mathbb{S} \to M\mathrm{U}_{\mathbb{R}} \to E_4$ where E_4 is the restriction to C_2-rings. Using that the norm is the adjoint to the restriction, this norms up to a map $\mathbb{S} \to M\mathrm{U}^{((C_8))} = N_{C_2}^{C_8} M\mathrm{U}_{\mathbb{R}} \to E_4$. Taking C_8 homotopy fixed points gives $\mathbb{S} \to \left(M\mathrm{U}^{((C_8))}\right)^{hC_8} \to E_4^{hC_8}$ so this will hopefully suffice to detect θ_j for $j \geq 7$.

However, there is no reason to believe that this spectrum will be periodic. To get that, one has to invert something. Indeed, they carefully find some analogue of the Bott class $D : \mathbb{S}^{\ell\rho_8} \to M\mathrm{U}^{((C_8))}$ (where $\rho_8 = \mathbb{R}[C_8]$), which will make it periodic and won't ruin the other properties. Then, they define the C_8 genuine spectrum $\Omega_{\mathbb{O}} = D^{-1} M\mathrm{U}^{((C_8))}$, and $\Omega = \Omega_{\mathbb{O}}^{hC_8}$ is finally the desired spectrum.

Everything above was just a motivation, if you lost me, here's the bottom line:

1. Construct $M\mathrm{U}_{\mathbb{R}}$,
2. Norm up to C_8 to get $M\mathrm{U}^{((C_8))} = N_{C_2}^{C_8} M\mathrm{U}_{\mathbb{R}}$,
3. Invert something to get $\Omega_{\mathbb{O}} = D^{-1} M\mathrm{U}^{((C_8))}$ (we will not talk about D today too much),
4. Take $\Omega = \Omega_{\mathbb{O}}^{hC_8}$.

3 Construction of $M\mathrm{U}_{\mathbb{R}}$

We now construct $M\mathrm{U}_{\mathbb{R}}$, as a very simple application of the theory we developed in the first part of the seminar. First, we recall that using the fiber bundles $\gamma_n^{\mathrm{U}} : E\mathrm{U}(n) \to B\mathrm{U}(n)$, $\gamma_n^{\mathrm{O}} : E\mathrm{O}(n) \to B\mathrm{O}(n)$ and their Thom spaces, we can construct the ring spectra $M\mathrm{U}$ and $M\mathrm{O}$. Now we do the two constructions at the same time.

C_2 has two irreducible representations over $\mathbb{R}$, both of them one dimensional, and we denote them $1, \alpha$. Consider the action of C_2 on $\mathbb{C}$ by conjugation (not $\mathbb{C}$-linear), with fixed points $\mathbb{R}$. This C_2-representation is $\rho = 1 + \alpha$.

Via this action, C_2 acts on $\mathrm{U}(n)$ with fixed points $\mathrm{O}(n)$, i.e. we get a C_2-group which we denote by $\mathbb{U}(n)$ that has $\underline{\mathbb{U}(n)} = \mathrm{U}(n)$ and $\mathbb{U}(n)^{C_2} = \mathrm{O}(n)$. In this world we can now form the classifying space $\gamma_n^{\mathbb{U}} : E\mathbb{U}(n) \to B\mathbb{U}(n)$ (e.g. by bar construction). Now we can take $\mathrm{Th}\left(\gamma_n^{\mathbb{U}}\right) \in \mathcal{S}^{O_{C_2}^{\mathrm{op}}}$. Now, $\mathrm{rk}_\rho\left(\gamma_n^{\mathbb{U}} \oplus \rho\right) = n+1$, so this bundle is a pullback of $\gamma_{n+1}^{\mathbb{U}}$, i.e. there is a map $f_n : B\mathbb{U}(n) \to B\mathbb{U}(n+1)$ (actually just the B of the map that takes a matrix and adds a 1 at the end of the diagonal) such that $\gamma_n^{\mathbb{U}} \oplus \rho = f_n^* \gamma_{n+1}^{\mathbb{U}}$. This gives a map $\Sigma^\rho \mathrm{Th}\left(\gamma_n^{\mathbb{U}}\right) = \mathrm{Th}\left(\gamma_n^{\mathbb{U}} \oplus \rho\right) \to \mathrm{Th}\left(\gamma_{n+1}^{\mathbb{U}}\right)$. This motivates normalizing as follows, using the functor $\Sigma_{C_2}^\infty : \mathcal{S}^{O_{C_2}^{\mathrm{op}}} \to \mathrm{Sp}_{C_2}$, define $M\mathrm{U}_{\mathbb{R}}(n) = \Sigma^{-\rho n} \Sigma_{C_2}^\infty \mathrm{Th}\left(\gamma_n^{\mathbb{U}}\right)$, and the above map gives us $M\mathrm{U}_{\mathbb{R}}(n) \to M\mathrm{U}_{\mathbb{R}}(n+1)$. We denote the colimit by $M\mathrm{U}_{\mathbb{R}} = \mathrm{colim}\, M\mathrm{U}_{\mathbb{R}}(n)$. Moreover, the multiplication maps $\mathbb{U}(n) \times \mathbb{U}(m) \to \mathbb{U}(n+m)$, and the identity $1 \to \mathbb{U}(n)$, make $M\mathrm{U}_{\mathbb{R}}$ into a ring.

Note that by construction $\underline{\mathrm{Th}\left(\gamma_n^{\mathbb{U}}\right)} = \mathrm{Th}\left(\gamma_n^{\mathrm{U}}\right)$ and $\mathrm{Th}\left(\gamma_n^{\mathbb{U}}\right)^{C_2} = \mathrm{Th}\left(\gamma_n^{\mathrm{O}}\right)$. Recall that Lior told us that for $X \in \mathcal{S}^{O_G^{\mathrm{op}}}$ we have $\Phi^H\left(\Sigma_G^\infty X\right) = \Sigma^\infty X^H$, thus we get that $\underline{M\mathrm{U}_{\mathbb{R}}} = \Phi^1 M\mathrm{U}_{\mathbb{R}} = M\mathrm{U}$ and $\Phi^{C_2} M\mathrm{U}_{\mathbb{R}} = M\mathrm{O}$.

4 Construction of $M\mathrm{U}^{((C_8))}$

We define $M\mathrm{U}^{((C_8))} = N_{C_2}^{C_8} M\mathrm{U}_{\mathbb{R}}$, so let's recall what the norm is. In fact, $M\mathrm{U}_{\mathbb{R}}$ is a genuine C_2-ring spectrum and we want $M\mathrm{U}^{((C_8))}$ to be a genuine C_8-ring spectrum.

Generally, let $H \leq G$ be a subgroup, and consider the forgetful $\mathrm{CAlg}\left(\mathrm{Sp}_H\right) \to \mathrm{CAlg}\left(\mathrm{Sp}_G\right)$. Its left adjoint is the norm $N_H^G : \mathrm{CAlg}\left(\mathrm{Sp}_H\right) \to \mathrm{CAlg}\left(\mathrm{Sp}_G\right)$ (in the sense that the underlying G-spectrum of this norm, is the norm (of spectra, not rings) of the underlying H-spectrum).

In our case, $M\mathrm{U}^{((C_8))}$ can be thought of as the spectrum $M\mathrm{U}_{\mathbb{R}} \otimes M\mathrm{U}_{\mathbb{R}} \otimes M\mathrm{U}_{\mathbb{R}} \otimes M\mathrm{U}_{\mathbb{R}}$, where the generator of C_8 acts by $(a, b, c, d) \mapsto \left(\overline{d}, a, b, c\right)$.

5 Idea of the Proof, Again

Recall that the main theorem follows easily from the *Detection*, *Gap* and *Periodicity* theorems. We now refine them.

The proof of the Gap Theorem actually splits into two parts. Recall that we have defined $\Omega = \Omega_{\mathbb{O}}^{hC_8}$, but $\Omega_{\mathbb{O}}$ is a *genuine* C_8 spectrum, therefore we can also consider the declared fixed points which map to the homotopy fixed points $\Omega_{\mathbb{O}}^{C_8} \to \Omega_{\mathbb{O}}^{hC_8}$. This is where we use the power of genuine G-spectra. The original Gap theorem follows easily from the following two results:

Theorem (Homotopy Fixed Point)**.** *The map* $\Omega_{\mathbb{O}}^{C_8} \xrightarrow{\sim} \Omega_{\mathbb{O}}^{hC_8}$ *is an equivalence.*

Theorem (Gap)**.** $\pi_{-2}\left(D^{-1} M\mathrm{U}^{((C_8))}\right)^{C_8} = 0$ *for any* $D : \mathbb{S}^{\ell\rho_8} \to M\mathrm{U}^{((C_8))}$ *(in fact* π_{-1}, π_{-3} *vanish as well).*

These two theorems and the Periodicity Theorem are proved by the slice spectral sequence (although the former considers the declared fixed points and the latter the homotopy fixed points, which are, as we said, equivalent). The proof of the Homotopy Fixed Point, Periodicity and Detection theorems depend crucially on the choice of D, most conditions coming from the Detection theorem.

The Detection Theorem is proved directly from the *Algebraic Detection Theorem*, which we now turn to.

6 The Detection Theorem

Recall that we want to prove that if θ_j exists, its image in $\pi_{2^{j+1}-2}\Omega$ is non-zero. How can we show something like that? One possible way, is using spectral sequences. Recall that we have seen in Shaul's talk (Browder) that θ_j exists iff $h_j^2 \in E_2^{2,2^{j+1}}(\mathbb{S}, H\mathbb{F}_2)$ is a permanent cycle (supporting θ_j). We have a spectral sequence computing the homotopy groups of $\Omega = \Omega_{\mathbb{O}}^{hC_8}$, namely the homotopy fixed point spectral sequence (which we will review soon). The trouble is that there is a priori no map between those spectral sequences, although we have a map $\mathbb{S} \to \Omega$. To handle this, we form a *span of spectral sequences.*

$$\begin{array}{ccccc} H\mathbb{F}_2\text{-based ASS of } \mathbb{S} & \longleftarrow & M\mathrm{U}\text{-based ANSS of } \mathbb{S} & \longrightarrow & C_8 \text{ homotopy fixed point SS of } \Omega_{\mathbb{O}} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \pi_*\mathbb{S} & = & \pi_*\mathbb{S} & \longrightarrow & \pi_*\Omega \end{array}$$

Recall that Shachar showed us how to construct the $H\mathbb{F}_2$-based ASS for a spectrum X. The idea was to look at the resolution of X by the cosimplicial object $X \otimes H\mathbb{F}_2^{\otimes n+1}$, use the maps to build a filtration (Adams filtration) which gives rise to a spectral sequence. This construction can be carried with $H\mathbb{F}_2$ replaced by another ring spectrum E, which will then (under some conditions) converge to $\pi_* L_E X$. For example, this can be done for $M\mathrm{U}$, for which $L_{M\mathrm{U}}\mathbb{S} = \mathbb{S}$. Furthermore, $H\mathbb{F}_2$ is complex oriented, i.e. it admits a map (of rings!) $M\mathrm{U} \to H\mathbb{F}_2$, which clearly gives a map of cosimplicial objects $X \otimes H\mathbb{F}_2^{\otimes n+1} \to X \otimes M\mathrm{U}^{\otimes n+1}$. We specialize to the case $X = \mathbb{S}$, yielding $H\mathbb{F}_2^{\otimes n+1} \to M\mathrm{U}^{\otimes n+1}$.

Now, let A be G-spectrum, and say we want to compute the homotopy groups of A^{hG}. Recall that $A^{hG} = \mathrm{Map}^G(EG, A)$. Now, EG has a simplicial model, $EG^n = G^{n+1}$ (i.e. $|EG^\bullet|$ is a model for EG). This gives us a cosimplicial object $C^n(G; A) = \mathrm{Map}^G(EG^n, A) \cong \prod_{G^n} A$, which gives a resolution of A^{hG}. The associated spectral sequence has E_2 page given by $E_2^{s,t}(G, A) = H^s(G; \pi_t A)$, and converges to $\pi_{t-s}A^{hG}$. Note that $C^0(G; A) = A$. We specialize to $C^n(C_8, \Omega_{\mathbb{O}})$.

Now, $\Omega_{\mathbb{O}}$ is complex orientable, i.e. admits a map $M\mathrm{U} \to \Omega_{\mathbb{O}} = C^0(C_8, \Omega_{\mathbb{O}})$ (because it was constructed from it). We have $n+1$ maps $C^0(C_8, \Omega_{\mathbb{O}}) \to C^n(C_8, \Omega_{\mathbb{O}})$, the i-th of which corresponding to the map $[0] \to [n]$ going to the

i-th vertex. Use this to define the i-th coordinate of $M\mathrm{U}^{\otimes n+1} \to C^n(C_8, \Omega_{\mathbb{O}})$. This gives the second map of spectral sequences. Conceptually, this map corresponds to the inclusion of the single point $\pi_*\Omega_{\mathbb{O}}$ with C_8-action, in the moduli stack of formal group laws.

Now, we claim that the following will imply the Detection Theorem.

Theorem (Algebraic Detection)**.** *Let $x \in E_2^{2,2^{j+1}}(\mathbb{S}, M\mathrm{U})$ be an element mapping to $h_j^2 \in E_2^{2,2^{j+1}}(\mathbb{S}, H\mathbb{F}_2)$, then its image $b_j \in H^2(C_8, \pi_{2^{j+1}}\Omega_{\mathbb{O}})$ is non-zero.*

Proof of Detection Theorem. Assume that θ_j exists, then by Browder, it is supported by h_j^2. Moreover, it must be supported by some element in $x \in E_2^{*,*}(\mathbb{S}, M\mathrm{U})$. The Adams filtration can only decrease by maps, and $E_2^{0,*}(\mathbb{S}, M\mathrm{U}), E_2^{1,*}(\mathbb{S}, M\mathrm{U}) = 0$ by classical computations, so $x \in E_2^{2,2^{j+1}}(\mathbb{S}, M\mathrm{U})$, mapping to h_j^2. Therefore, by the Algebraic Detection Theorem, its image $b_j \in H^2(C_8, \pi_{2^{j+1}}\Omega_{\mathbb{O}})$ is non-zero. The only differential that can hit it, is $d_2 : H^0(C_8, \pi_{2^{j+1}-1}\Omega_{\mathbb{O}}) \to H^2(C_8, \pi_{2^{j+1}}\Omega_{\mathbb{O}})$ (because the others are negative cohomology groups). Note that $\Omega_{\mathbb{O}}$ was constructed from $M\mathrm{U}_{\mathbb{R}}$ which is even, and taking the norm and inverting D preserve that, thus the source of the d_2 is 0, so b_j is a permanent cycle. This means that the image of θ_j in $\pi_{2^{j+1}-2}\Omega$ is indeed non-zero. □

References

[HHR09] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. "On the non-existence of elements of Kervaire invariant one". In: *arXiv:0908.3724* (2009).

[HHR10] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. "The Arf-Kervaire problem in Algebraic Topology: Sketch of the proof". In: *https://web.math.rochester.edu/people/faculty/doug/mypapers/CDM%20Proof6.pdf* (2010).

[Mil11] Haynes Miller. "Kervaire Invariant One [after M. A. Hill, M. J. Hopkins, and D. C. Ravenel]". In: *arXiv:1104.4523* (2011).

[Sch07] Stefan Schwede. "An untitled book project about symmetric spectra". In: *http://www.math.uni-bonn.de/~schwede/SymSpec.pdf* (2007).

[Tal16] Talbot. "Equivariant stable homotopy theory and the Kervaire invariant". In: *https://math.mit.edu/conferences/talbot/index.php?year=2016* (2016).