Caesarea Workshop Preparation 2019
HKR Generalized Character Theory

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1 Character Theory and HKR

Let G be a finite group. We recall that the basic construction of character theory is a map $\chi : \mathrm{Rep}\,(G) \to \prod_{G/G} C_0$, where $\mathrm{Rep}\,(G)$ is the representation ring, and $C_0 = \mathrm{colim}\,\mathbb{Q}\,(\zeta_m) = \mathbb{Q}\,(\zeta_\infty)$. One of the main theorems is then

Theorem 1. *The character map is an isomorphism when tensored with C_0, that is $C_0 \otimes \mathrm{Rep}\,(G) \xrightarrow{\sim} \prod_{G/G} C_0$.*

Let's reconstruct this map. First, conjugating a representation gives an isomorphic representation, that is the same element in the representation ring. Therefore conjucation induces identity on $\mathrm{Rep}\,(G)$, so we can factor $\mathrm{Rep} : \mathrm{FinGrp}^{\mathrm{op}} \xrightarrow{\mathrm{B}} \mathcal{S}^{\mathrm{op}} \to \mathrm{CRing}$. Define the *lattice* as the formal system (do not take the limit) $\mathbb{L} = \lim \mathbb{Z}/m$. We define the *(formal) free loop space* of $\mathrm{B}G$ to be $\mathcal{L}\mathrm{B}G = \mathrm{Map}\,(\mathrm{B}\mathbb{L}, \mathrm{B}G)$ (if $\mathbb{L}$ were $\mathbb{Z}$ it was really the free loop space). We get a map $\pi_0\mathcal{L}\mathrm{B}G \to \hom\,(\mathrm{Rep}\,(\mathrm{B}G)\,, \mathrm{Rep}\,(\mathrm{B}\mathbb{L}))$, that is $\mathrm{Rep}\,(G) \to \prod_{\pi_0\mathcal{L}\mathrm{B}G} \mathrm{Rep}\,(\mathbb{L})$. We have an algebraic map (indepedent of G) given by $\mathrm{Rep}\,(\mathbb{L}) = \mathrm{colim}\,\mathbb{Z}\,[t]\,/\,(t^m - 1) \to \mathrm{colim}\,\mathbb{Q}\,(\zeta_m) = \mathbb{Q}\,(\zeta_\infty) = C_0$. We compose to get

$$\begin{aligned} \mathrm{Rep}\,(G) &\to \prod_{\pi_0\mathcal{L}\mathrm{B}G} \mathrm{Rep}\,(\mathbb{L}) && \rho \rightsquigarrow \alpha^*\rho\,(i) = \rho\left(g^i\right) \\ &\to \prod_{\pi_0\mathcal{L}\mathrm{B}G} C_0 && \rightsquigarrow \mathrm{tr}\,(\alpha^*\rho\,(1)) = \mathrm{tr}\,(\rho\,(g)) \\ &\overset{(\star)}{=} \mathrm{H}C_0^0\,(\mathcal{L}\mathrm{B}G) \end{aligned}$$

To connect this to the usual description we note that $\mathcal{L}\mathrm{B}G = \mathrm{Map}\,(\mathrm{B}\mathbb{L}, \mathrm{B}G) = \mathrm{Map}_*\,(\mathrm{B}\mathbb{L}, \mathrm{B}G)\,//G = \hom\,(\mathbb{L}, G)\,//G = G//G$, thus $\pi_0\mathcal{L}\mathrm{B}G = G/G$. In $(\star)$ we use that each connected component is $\mathrm{B}H$, thus rationally trivial.

Recall from Nat's talk that Atiyah-Segal theorem tells us that $K\,(\mathrm{B}G) = \mathrm{Rep}\,(G)^\wedge_I$, thus $K\,(\mathrm{B}G)$ can be studied using characters. As we know, there are higher analogues of (p-complete) K-theory, namely Morava E-theory $E = E_n$ of height $n < \infty$ at the prime p. Thus, $E^0(\mathrm{B}G)$ should be though of as some generalized representation ring, and our goal is to study it, or rather its rationalization, using something similar to the character map (it should be noted that everything is correct at the graded setting $E^*(\mathrm{B}G)$ as well, we restrict to E^0 to keep the exposition simple).

Similarly to before (only this time we are in a p-complete situation), define the *lattice*, to be the formal system $\mathbb{L} = \lim (\mathbb{Z}/p^r)^n$ (whose limit is $\mathbb{Z}_p^n$). Also, we define the *(formal) n-free loop space* $\mathcal{L}^n\mathrm{B}G = \mathrm{Map}(\mathrm{B}\mathbb{L}, \mathrm{B}G)$. Like before, $\mathcal{L}^n\mathrm{B}G = \hom(\mathbb{L}, G)//G = G_p^{(n)}//G$ where $G_p^{(n)}$ is the set of tuples of n elements of order power of p, acted by conjugation.

In the classical case we had to take the effect of Rep on mapping space and use the exponential rule, topologically this can be done at once. Then, we use some algebraic map like in the classical case.

Topological part - Consider the evaluation map $\mathrm{B}\mathbb{L}\times\mathcal{L}^n\mathrm{B}G = \mathrm{B}\mathbb{L}\times\mathrm{Map}(\mathrm{B}\mathbb{L}, \mathrm{B}G) \to \mathrm{B}G$. Taking E^0 we get $E^0(\mathrm{B}G) \to E^0(\mathrm{B}\mathbb{L} \times \mathcal{L}^n\mathrm{B}G) \cong E^0(\mathrm{B}\mathbb{L})\otimes_{E^0} E^0(\mathcal{L}^n\mathrm{B}G)$, where Kunneth holds because (each part of the diagram) $E^0(\mathrm{B}\mathbb{L})$ is a free E^0-module.

Algebraic part - Later on we will construct a map $E^0(\mathrm{B}\mathbb{L}) \to C_0$, where C_0 is some (flat) $p^{-1}E^0$-algebra, and in the case $n = 1$, $C_0 = \mathbb{Q}_p(\zeta_{p^\infty})$. Further, we have a map of spectra $E \to p^{-1}E$.

Combining these we get the following map

$$\begin{aligned}\chi : E^0(\mathrm{B}G) &\xrightarrow{\text{topological}} E^0(\mathrm{B}\mathbb{L}) \otimes_{E^0} E^0(\mathcal{L}^n\mathrm{B}G)\\ &\xrightarrow{\text{algebraic}} C_0 \otimes_{p^{-1}E^0} \left(p^{-1}E\right)^0(\mathcal{L}^n\mathrm{B}G)\\ &= \mathrm{H}C_0^0(\mathcal{L}^n\mathrm{B}G)\end{aligned}$$

Like in the original case, we can identify the last term with $\prod_{G_p^{(n)}/G} C_0$, using that $\mathcal{L}^n\mathrm{B}G = G_p^{(n)}//G$ and the rationality. Then indeed in the case $n = 1$, i.e. $E = K_p^\wedge$, at least for a p-group, it recovers some completed form of the usual character map, and for higher n gives some generalized characters.

In fact, by a small modification, we can generalize the definition (which will be useful later even to prove the case above) to incorporate a G-space X. It may seem as though we can simply replace each instance of $\mathrm{B}G$ by $X//G$, then $X = G/H$ gives the case of $(G/H)//G = \mathrm{B}H$. However, the mapping space $\mathrm{Map}(\mathrm{B}\mathbb{L}, X//G)$ in spaces doesn't take into account the genunie G structure on X. One way to solve it, is to use another category in which $\mathrm{B}G$, and $X//G$, live, namely the category of topological groupoids (this is a global situation). Here, $\mathrm{B}G \in \mathrm{TopGrpd}$ has topological space of objects $*$, and morphisms G. Similarly, for X we define $X//G$ to be the topological groupoid with objects X, and morphisms $X \times G$, i.e. there is a morphism $x \to gx$. We then define the *(formal) n-free loop space* by $\mathcal{L}^n(X//G) = \mathrm{Map}_{\mathrm{TopGrpd}}(\mathrm{B}\mathbb{L}, X//G)$, and we can then repeat the construction from above to obtain the general character map

$$\begin{aligned}\chi : E^0(X//G) &\xrightarrow{\text{topological}} E^0(\mathrm{B}\mathbb{L}) \otimes_{E^0} E^0(\mathcal{L}^n(X//G)) \\ &\xrightarrow{\text{algebraic}} C_0 \otimes_{p^{-1}E^0} \left(p^{-1}E\right)^0 (\mathcal{L}^n(X//G)) \\ &= \mathrm{HC}_0^0(\mathcal{L}^n(X//G))\end{aligned}$$

The main theorem of this talk is the following:

Theorem 2 ([HKR, theorem C] and [Pet, theorem A.1.23])**.** *The character map is an isomorphism when tensored with C_0 for every finite G-space X, that is $C_0 \otimes_{E^0} E^0(X//G) \xrightarrow{\sim} \mathrm{HC}_0^0(\mathcal{L}^n(X//G))$. In particular, for $X = G/G = *$, $C_0 \otimes_{E^0} E^0(\mathrm{B}G) \xrightarrow{\sim} \prod_{G_p^{(n)}/G} C_0$.*

2 Strategy of the Proof

First, we haven't defined C_0 yet. Second, one may wonder how adding finite G-spaces X helps. Here's the strategy:

1. Construct the ring C_0, such that $C_0 \otimes_{E^0} E^0(\mathrm{B}A) \xrightarrow{\sim} \prod_{A_p^n} C_0$ for abelian groups A (indepedent of G), this implies the theorem for $X = G/A$
2. Both functors commute with finite (homotopy) colimits, so we deduce the theorem for X with *abelian stabilizers*
3. Reduce from general G-spaces to G-spaces with abelian stabilizers using *complex oriented descent*

3 The Ring C_0

We construct C_0, such that the theorem is true for abelian groups. In this case, what we need is $C_0 \otimes_{E^0} E^0(\mathrm{B}A) \xrightarrow{\sim} \prod_{A_p^n} C_0$ ($A_p^n/A = A_p^n$ because A is abelian). For height 1 we should recover $C_0 = \mathbb{Q}_p(\zeta_{p^\infty})$. We take an algebro-geometric approach, and study the Spec of the the map.

The target is $\mathrm{Spec}\left(\prod_{A_p^n} C_0\right) = \underline{A_p^n}$ i.e. the constant group scheme A_p^n over C_0. Also, recall that Nat told us that $\mathrm{Spec}\, E^0(\mathrm{B}A) = \underline{\mathrm{GrpSch}}_{E^0}\left(\underline{\widehat{A}}, \mathbb{G}\right)$ where $\widehat{A} = \hom\left(A, S^1\right)$ is the Pontryagin-dual (for example, we get that $\mathrm{Spec}\, E^0(\mathrm{B}\mathbb{Z}/p^r) = \mathbb{G}[p^r] = \underline{\mathrm{GrpSch}}_{E^0}\left(\underline{\widehat{\mathbb{Z}/p^r}}, \mathbb{G}\right)$). What this actually means is that for any E^0-algebra R, we have $\hom_{E^0}\left(E^0(\mathrm{B}A), R\right) = \hom_{\mathrm{Grp}}\left(\widehat{A}, \mathbb{G}(R)\right)$. Therefore, the

Spec of the character map over C_0 is:

$$\underline{\text{GrpSch}}_{C_0}\left(\widehat{\underline{A}}, C_0 \otimes_{E^0} \mathbb{G}\right) \leftarrow \underline{A_p^n}$$

And we want to make this into an isomorphism. The solution is to require that $C_0 \otimes_{E^0} \mathbb{G} \cong \widehat{\underline{\mathbb{L}}}$ (again, we mean the diagrams $C_0 \otimes_{E^0} \mathbb{G}[p^r] \cong \underline{\widehat{\mathbb{L}/p^r}}$), then:

$$\underline{\text{GrpSch}}_{C_0}\left(\widehat{\underline{A}}, C_0 \otimes_{E^0} \mathbb{G}\right) \cong \underline{\text{GrpSch}}_{C_0}\left(\widehat{\underline{A}}, \widehat{\underline{\mathbb{L}}}\right) = \underline{\text{GrpSch}}_{C_0}\left(\underline{\mathbb{L}}, \underline{A}\right) = \underline{A_p^n}$$

The only question remaining, is how would we come up with such a $p^{-1}E^0$-algebra C_0. The solution is in two steps, first find an intermediate algebra over which we have a canonical map $u : \widehat{\underline{\mathbb{L}}} \to \mathbb{G}$, second take the universal place over which u becomes an isomorphism. (Also this is a classical situation of "there is a map and they are isomorphic", we don't have time to connect the two.)

For the first step, we use again (for a different reason!) the algebro-geometric interpretation of $E^0(\text{B}A)$. Taking $A = \mathbb{L} = \lim (\mathbb{Z}/p^r)^n$ we get $\text{Spec}\, E^0(\text{B}\mathbb{L}) = \underline{\text{GrpSch}}_{E^0}\left(\widehat{\underline{\mathbb{L}}}, \mathbb{G}\right)$. Thus, we take the intermediate algebra to be $E^0(\text{B}\mathbb{L})$, over which we have a canonical map $u : \widehat{\underline{\mathbb{L}}} \to E^0(\text{B}\mathbb{L}) \otimes_{E^0} \mathbb{G}$ (since it corepresents the scheme of such maps).

I remind you that what we really mean is the colimit over r of $u_r : \underline{\widehat{(\mathbb{Z}/p^r)^n}} \to E^0((\text{B}\mathbb{Z}/p^r)^n) \otimes_{E^0} \mathbb{G}[p^r]$. We now wish to make u into an isomorphism, and this is done level-wise. By taking global sections we get a map $\mathcal{O}(u_r)$ between two free $E^0((\text{B}\mathbb{Z}/p^r)^n)$-algebras of the same rank (p^{rn}). Thus to make u_r an isomorphism, it is enough to invert $\det \mathcal{O}(u_r)$, so define $C_0 = \text{colim}\,(\det \mathcal{O}(u_r))^{-1} E^0((\text{B}\mathbb{Z}/p^r)^n)$.

As we said, by construction, over C_0, the character map is an isomorphism for abelian groups. Furthermore, it turns out (see [HKR, lemma 6.3] and [Pet, lemma A.1.4]) that C_0 is a faithfully flat $p^{-1}E^0$-algebra, thus it preserves the rational information.

We don't have time to do the case $n = 1$ in detail. Let me just say that in this case $K_p^\wedge(\text{B}\mathbb{Z}/p^r) = \mathcal{O}(\mathbb{G}[p^r]) = \mathbb{Z}_p[t]/\left(t^{p^r} - 1\right)$, and inverting the determinant exactly induces the canonical map to $\mathbb{Q}_p(\zeta_{p^r})$, as we expect.

4 Complex Oriented Descent

Let me recap. We have generalized the character map to $\chi : E^0(X//G) \to \text{HC}_0^0(\mathcal{L}^n(X//G))$. For $X = G/H$, since $X//G = (G/H)//G = \text{B}H$, this reduces to the usual character map for the group H. We have shown the theorem for $X = G/A$. By the fact that the functors commute with finite colimits, the theorem holds for finite G-spaces X with abelian stabilizers. So, to finish all we need is to show that we can reduce from general finite G-spaces to finite G-spaces with abelian stabilizers. This is done using complex oriented descent.

Definition 3. Let V be a vector space of $\dim V = m$. Choose an inner product on V (contractible). Define the space of *complete flags*, as the space of n-orthogonal lines $\mathrm{Flag}(V) = \{(\ell_1, \dots, \ell_m) \mid \ell_i \perp \ell_j\}$ (the topology is the subspace topology of $\mathbb{P}(V)^m$). Similarly, for a vector bundle $V \to X$, we can choose a metric (contractible), and repeat the construction fiber-wise, to form the *flag bundle* $\mathrm{Flag}(V) \to X$.

We don't have the time to delve into it, but a complex orientation gives control over flag bundles, similarly to the way it gives control for complex bundles. An instance of this idea is the following:

Proposition 4 ([HKR, proposition 2.4])**.** *Let F be a complex orientable spectrum (e.g. $E_n, \mathrm{H}C_0$), and $V \to X$ a vector bundle. Then $F^0(\mathrm{Flag}(V))$ is a free module over $F^0(X)$ (of rank $m!$).*

Using the map $\mathrm{Flag}(V) \to X$, we can form a simplicial resolution $X \leftarrow (\mathrm{Flag}(V))^{\bullet}$. Taking F^0 gives $F^0(X) \to F^0\left((\mathrm{Flag}(V))^{\bullet}\right)$, that is a cosimplicial resolution of $F^0(X)$. The previous proposition shows that in fact:

$$F^0(X) = \ker\left(F^0(\mathrm{Flag}(V)) \to F^0(\mathrm{Flag}(V) \times_X \mathrm{Flag}(V))\right)$$

A functor (not necessarily a cohomology theory) satisfying this is said to satisfy *complex oriented descent.* To see why it useful for our situation, fix some *faithful* representation V of G. Of course, G acts on $\mathrm{Flag}(V)$. We claim that the stabilizer of any point $(\ell_1, \dots, \ell_m)$ is abelian, this is just because being in the stabilizer means being diagonial w.r.t to this decomposition, so all points in the stabilizer are simultaneously diagonalizable, thus commute. For a G-space X, we get that $\mathrm{Flag}(V \times X) = \mathrm{Flag}(V) \times X$ has abelian stabilizers.

To use complex oriented descent in our case we need to know that $F^0(X) = E^0(X//G)$ and $F^0(X) = \mathrm{H}C_0^0(\mathcal{L}^n(X//G))$ satisfy complex oriented descent. Indeed HKR show (see [HKR, proposition 2.6]) that for a G-vector bundle $V \to X$, we have $\mathrm{Flag}(V//G) = \mathrm{Flag}(V)//G \to X//G$, thus $E^0(X//G)$ satisfies complex oriented descent. Also, they show that $\mathrm{Flag}(V)^A \to X^A$ is constructed from flag bundles over X^A, which implies that the same result holds for $\mathrm{H}C_0^0(\mathcal{L}^n(X//G))$.

Therefore, we have reduced from general G-spaces to G-spaces with abelian stabliziers, and the proof is concluded.

References

[HKR] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. *Generalized group characters and complex oriented cohomology theories.*

[Pet] Eric Peterson. *Formal Geometry and Bordism Operations.* URL: `https://github.com/ecpeterson/FormalGeomNotes`.