Caesarea Workshop 2018
A stratified homotopy hypothesis

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The paper `https://arxiv.org/abs/1502.01713`.

1 Intro

Throughout the talk stratified spaces are conically smooth stratified spaces, and maps between them are conically smooth. Their (1-)category is denoted by $\underline{\mathrm{Strat}}$.

The homotopy hypothesis is the assertion that spaces (up to equivalence) are a model for ∞-groupoids. The idea of this paper is that we can model ∞-categories using stratified spaces. The basic construction is the functor $\mathrm{Exit} : \mathcal{S}\mathrm{trat} \to \mathrm{Cat}_\infty$ (where $\mathcal{S}\mathrm{trat}$ is some ∞-category of stratified spaces) called the *exit path*, which is fully faithful. $\mathrm{Exit}(K)$ should be thought of as the category whose objects are points in K, morphisms are paths in K which once leave a stratum don't come back to it. This basic idea allows the non-invertibility of morphisms. This functor by itself, although fully faithful, doesn't model ∞-categories. We can then look at restricted Yoneda along Exit, and discover that for each $\mathcal{C} \in \mathrm{Cat}_\infty$ the presheaf $K \mapsto \mathrm{Map}(\mathrm{Exit}(K), \mathcal{C})$ satisfies some conditions. A presheaf satisfying these conditions is called a *striation sheaf*, and thus we arrive at a functor $\mathrm{Cat}_\infty \to \mathcal{S}\mathrm{tri}$, which will turn out to be an equivalence, thus we get a model of ∞-categories as certain sheaves on stratified spaces. This model is very geometric, thus allowing to construct many ∞-categories with geometric origins directly.

2 Constructible Sheaves and $\mathcal{S}\mathrm{trat}$

Definition 1. A *covering sieve* of $K \in \underline{\mathrm{Strat}}$ is a full subcategory $\mathcal{U}$ of $\underline{\mathrm{Strat}}_{/K}$ with the following properties:

1. *Sieve* - if $U \to K \in \mathcal{U}$ and $V \to U$ is a morphism in $\underline{\mathrm{Strat}}$, then $V \to U \to K \in \mathcal{U}$.

2. *Open* - each $U \to K \in \mathcal{U}$ can be factored as a $U \to U_0 \to K$ for $U_0 \to K \in \mathcal{U}$ open embedding.

3. *Surjective* - for each $x \in K$, $\{x\} \to K \in \mathcal{U}$.

Example 2. If $\bigcup U_\alpha = K$ is an open cover, then the collection of all maps $A \to U_\alpha \hookrightarrow K$ is a covering sieve of K.

Definition 3. The ∞-category of *sheaves* on $\underline{\mathrm{Strat}}$ is the full ∞-subcategory $\mathrm{Shv}(\underline{\mathrm{Strat}}) \subset \mathrm{PShv}(\underline{\mathrm{Strat}})$, of all presheaves $\mathcal{F}$ s.t. for each $K \in \underline{\mathrm{Strat}}$ and a covering sieve $\mathcal{U}$ of K, $\mathcal{F}(K) \to \lim_{U \in \mathcal{U}} \mathcal{F}(U)$ is an equivalence.

Remark 4. Since each stratified space K has a conically smooth atlas $\left\{\mathbb{R}^{i_\alpha} \times \mathrm{C}(Z_\alpha)\right\}$, which is in particular an open cover, we see that $\mathcal{F}$ is determined by its values on basics $\mathbb{R}^i \times \mathrm{C}(Z)$.

Note that a (usual) homotopy $X \times \mathbb{R} \to Y$ is the same as a map $X \times \mathbb{R} \to Y \times \mathbb{R}$ over $\mathbb{R}$.

Definition 5. Let $f, g : X \to Y$ be maps of stratified spaces. A *stratified homotopy* is $H : X \times \mathbb{R} \to Y \times \mathbb{R}$, which restrict to f resp. g at 0 resp. 1. We then denote $f \simeq g$. $f : X \to Y$ is a *stratified homotopy equivalence* if it has a stratified homotopy inverse. We denote by $\mathcal{J}$ the collection of all stratified homotopy equivalences, and by $\mathcal{R}$ the collection of all projections $X \times \mathbb{R}^n \to X$.

Lemma 6. *The following holds:*

1. *Stratified homotopy equivalence is an equivalence relation.*

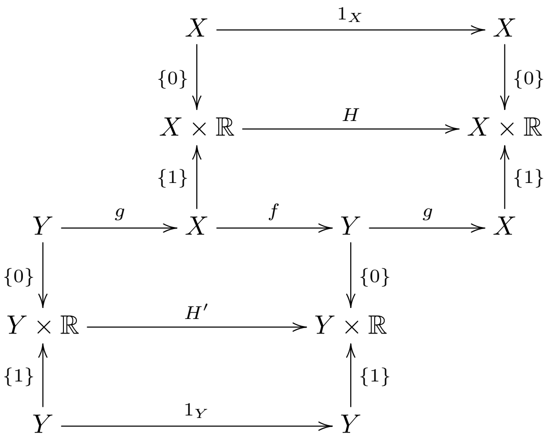
2. *Stratified homotopy equivalence satisfies 2-out-of-3.*

3. *The projection $X \times \mathbb{R}^n \to X$ is a stratified homotopy equivalence (i.e. $\mathcal{R} \subset \mathcal{J}$.)*

Lemma 7. *The canonical map $\underline{\mathrm{Strat}}\left[\mathcal{R}^{-1}\right] \to \underline{\mathrm{Strat}}\left[\mathcal{J}^{-1}\right]$ is an equivalence of ∞-categories.*

Proof. Let $F : \underline{\mathrm{Strat}} \to \mathcal{C}$ be a functor to an ∞-category $\mathcal{C}$, which carries morphisms in $\mathcal{R}$ to equivalences. For each stratified homotopy equivalence f :

$X \to Y$, there is a commutative diagram as follows:



Then since all vertical arrows are mapped by F to equivalences, it follows that $F(gf)$ and $F(fg)$ are homotopic to the identities, thus $F(f)$ is an equivalence.

$\square$

Definition 8 (/Lemma)**.** The ∞-category of *constructible sheaves on* $\underline{\mathrm{Strat}}$ is the full subcategory $\mathrm{Shv}^{\mathrm{cbl}}(\underline{\mathrm{Strat}}) \subset \mathrm{Shv}(\underline{\mathrm{Strat}})$ of the sheaves $\mathcal{F} : \underline{\mathrm{Strat}}^{\mathrm{op}} \to \mathcal{S}\mathrm{paces}$ that satisfy the following equivalent conditions:

1. $\mathcal{F}$ factors through $\underline{\mathrm{Strat}}\left[\mathcal{J}^{-1}\right]^{\mathrm{op}}$.
2. $\mathcal{F}$ factors through $\underline{\mathrm{Strat}}\left[\mathcal{R}^{-1}\right]^{\mathrm{op}}$.
3. For each stratified homotopy equivalence $X \to Y$, $\mathcal{F}(Y) \to \mathcal{F}(X)$ is an equivalence.
4. For each stratified space K, $\mathcal{F}(K) \to \mathcal{F}(K \times \mathbb{R})$ is an equivalence.

The inclusion has a left adjoint $\mathrm{Shv}^{\mathrm{cbl}}(\underline{\mathrm{Strat}}) \underset{L}{\rightleftarrows} \mathrm{Shv}(\underline{\mathrm{Strat}})$. It can be proved that the Yoneda functor $\underline{\mathrm{Strat}} \to \mathrm{PShv}(\underline{\mathrm{Strat}})$ actually factors through (isotopy sheaves) $\underline{\mathrm{Strat}} \to \mathrm{Shv}(\underline{\mathrm{Strat}})$. We then compose with the functor L, to get $\underline{\mathrm{Strat}} \to \mathrm{Shv}^{\mathrm{cbl}}(\underline{\mathrm{Strat}})$ given by $K \mapsto L\,\mathrm{hom}_{\underline{\mathrm{Strat}}}(-, K)$.

Definition 9. The ∞-category of *conically smooth stratified spaces* $\mathcal{S}\mathrm{trat}$ is the essential image of that functor.

Theorem 10. *The functor* $\underline{\mathrm{Strat}} \to \mathcal{S}\mathrm{trat}$ *induces an equivalence of* ∞*-categories* $\underline{\mathrm{Strat}}\left[\mathcal{J}^{-1}\right] \to \mathcal{S}\mathrm{trat}$.

Proof. We have the following commutative diagram

$$\begin{array}{ccc}\underline{\mathrm{Strat}} & \longrightarrow & \underline{\mathrm{Strat}}\left[\mathcal{J}^{-1}\right] \\ \big\downarrow & & \big\downarrow \\ \mathrm{Shv}\left(\underline{\mathrm{Strat}}\right) & \xrightarrow{L} & \mathrm{Shv}^{\mathrm{cbl}}\left(\underline{\mathrm{Strat}}\right)\end{array}$$

Since it commutes, the essential images of both paths are equal, and the right functor, being a Yoneda embedding, is fully faithful, thus an equivalence to its essential image. □

Remark 11. The description via constructible sheaves, using some more ideas, leads to a Kan-enriched model $\mathrm{Map}_{\mathcal{S}\mathrm{trat}}\left(X, Y\right)_n = \mathrm{hom}_{\underline{\mathrm{Strat}}}\left(X \times \Delta^n_e, Y\right)$, but we don't have time to discuss that.

The functor $\underline{\mathrm{Strat}} \to \mathcal{S}\mathrm{trat}$ induces an adjunction $\mathrm{PShv}\left(\underline{\mathrm{Strat}}\right) \rightleftarrows \mathrm{PShv}\left(\mathcal{S}\mathrm{trat}\right)$ by pullback and right Kan extension.

Definition 12. The ∞-category of *sheaves* on $\mathcal{S}\mathrm{trat}$ is the pullback

$$\begin{array}{ccc}\mathrm{Shv}\left(\mathcal{S}\mathrm{trat}\right) & \longrightarrow & \mathrm{PShv}\left(\mathcal{S}\mathrm{trat}\right) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Shv}\left(\underline{\mathrm{Strat}}\right) & \longrightarrow & \mathrm{PShv}\left(\underline{\mathrm{Strat}}\right)\end{array}$$

It is then evident that we have:

Theorem 13. *The adjunction above restricts to an equivalence of ∞-categories* $\mathrm{Shv}^{\mathrm{cbl}}\left(\underline{\mathrm{Strat}}\right) \cong \mathrm{Shv}\left(\mathcal{S}\mathrm{trat}\right)$.

3 Exit Paths

Remember how for a space X we define $\mathrm{Sing}\left(X\right)$ as a complete Segal space by the composition $\mathrm{Sing} : \mathcal{S}\mathrm{paces} \to \mathrm{PShv}\left(\mathcal{S}\mathrm{paces}\right) \to \mathrm{PShv}\left(\Delta\right)$ given by $\mathrm{Sing}\left(X\right)\left([n]\right) = \mathrm{Map}\left(\Delta^n, X\right)$. That is we are using the cosimplicial object $\Delta \to \mathcal{S}\mathrm{paces}$ given by $[n] \mapsto \Delta^n$ to do restricted Yoneda. We would like to do something similar, but for stratified spaces.

Definition 14. The *standard* cosimplicial stratified space is $\mathrm{st} : \Delta \to \underline{\mathrm{Strat}}$ given on objects by $\mathrm{st}\left([n]\right) = \Delta^n \to [n]$ where the stratification is $\left(t_i\right) \mapsto \max\left\{i \mid t_i \neq 0\right\}$. Note that this is also given by $\overline{\mathrm{C}}^n\left(*\right)$. We denote the composition of st with $\underline{\mathrm{Strat}} \to \mathcal{S}\mathrm{trat}$ by $\mathrm{st} : \Delta \to \mathcal{S}\mathrm{trat}$ as well.

Definition 15. The *exit path* functor $\mathrm{Exit} : \mathcal{S}\mathrm{trat} \xrightarrow{\mathrm{Yoneda}} \mathrm{PShv}\left(\mathcal{S}\mathrm{trat}\right) \stackrel{\mathrm{st}^*}{\to} \mathrm{PShv}\left(\Delta\right)$ is defined as the restricted Yoneda. On objects it is given by $\mathrm{Exit}\left(X\right)\left([n]\right) = \mathrm{Map}_{\mathcal{S}\mathrm{trat}}\left(\Delta^n, X\right)$ (where Δ^n is with the standard stratification.)

Claim 16. For each compact stratified space L the following diagram in $\mathcal{S}\mathrm{trat}$ is a pushout:

$$\begin{array}{ccc} \overline{\mathrm{C}}(\emptyset) & \xrightarrow{\overline{\mathrm{C}}(\emptyset \hookrightarrow L)} & \overline{\mathrm{C}}(L) \\ {\scriptstyle\{1\}}\downarrow & & \downarrow{\scriptstyle\{1\}} \\ \overline{\mathrm{C}}^2(\emptyset) & \xrightarrow{\overline{\mathrm{C}}^2(\emptyset \hookrightarrow L)} & \overline{\mathrm{C}}^2(L) \end{array}$$

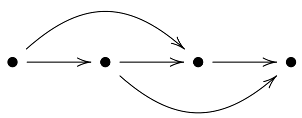
Lemma 17. *For each $X \in \mathcal{S}\mathrm{trat}$, $\mathrm{Exit}\,(X)$ is a complete Segal space.*

Proof. For $n \geq 2$ take $L = \Delta^{n-2}$ in the claim above to get the pushout:

$$\begin{array}{ccc} \Delta^{[1:1]} & \longrightarrow & \Delta^{[1:n]} \\ \downarrow & & \downarrow \\ \Delta^{[0:1]} & \longrightarrow & \Delta^{[0:n]} \end{array}$$

Now, since $\mathrm{Exit}\,(X)\,([n]) = \mathrm{Map}_{\mathcal{S}\mathrm{trat}}\,(\Delta^n, X)$, and the map functor sends colimits in the first coordinate to limits, we get the pullback $\mathrm{Exit}\,(X)\,([n]) = \mathrm{Exit}\,(X)\,([1:n]) \times_{\mathrm{Exit}(X)([1:1])} \mathrm{Exit}\,(X)\,([0:1])$, thus we see by induction that the Segal condition is satisfied.

The completeness condition is equivalent to the 2-out-of-6 property. Consider morphisms as follows:



then the 2-out-of-6 property is that if the two curved arrows are equivalence, then so are the other 4 (the 3 drawn, and the one that is their composition). Note that in our case, for each equivalence, we have a retract, and retracts must be from a stratum to itself. But just as in spaces, a path in a single stratum is invertible, so all retracts are invertible, and the 2-out-of-6 property follows.

Formally 2-out-of-6 is the claim that the following diagram is carried by $\mathrm{Map}_{\mathrm{PShv}(\Delta)^{\mathrm{op}}}\,(-, \mathrm{Exit}\,(X))$ to a pullback:

$$\begin{array}{ccc} \Delta\,[\{0 < 2\}] \coprod \Delta\,[\{1 < 3\}] & \longrightarrow & \Delta\,[\{0 < 1 < 2 < 3\}] \\ \downarrow & & \downarrow \\ \Delta\,[0] \coprod \Delta\,[0] & \longrightarrow & \Delta\,[0] \end{array}$$

$\square$

4 Striation Sheaves

st^* has a right adjoint $\mathrm{PShv}(\mathcal{S}\mathrm{trat}) \underset{\mathrm{st}_*}{\overset{\mathrm{st}^*}{\rightleftarrows}} \mathrm{PShv}(\Delta)$, given by $\mathrm{st}_*\,\mathcal{F}(X) = \mathrm{Map}(\mathrm{Exit}(X), \mathcal{F})$. We have seen before that $\mathrm{Shv}^{\mathrm{cbl}}(\underline{\mathrm{Strat}}) \cong \mathrm{Shv}(\mathcal{S}\mathrm{trat})$.

Definition 18. A presheaf $\mathcal{F} \in \mathrm{PShv}(\underline{\mathrm{Strat}})$ is called *cone-local* if for each compact stratified space L, $\mathcal{F}$ sends the following to a pullback:

$$\begin{array}{ccc} L & \longrightarrow & L \times \mathbb{R}_{\geq 0} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{C}(L) \end{array}$$

Through the equivalence above we claim:

Lemma 19. *The adjunction restricts to an equivalence* $\mathrm{Shv}^{\mathrm{cone,cbl}}(\underline{\mathrm{Strat}}) \cong \mathrm{PShv}(\Delta)$.

Proof. It can be verified directly that $\mathrm{st} : \Delta \to \mathcal{S}\mathrm{trat}$ is fully faithful, implying that st_* is fully faithful as well. It therefore remains to show that the unit $\mathcal{F} \to \mathrm{st}_*\,\mathrm{st}^*\,\mathcal{F}$ is an equivalence iff $\mathcal{F}$ is a cone-local constructible sheaf.

It is first proved that st_* takes values in cone-local constructible sheaf, showing the only if part. The sheaf and cone-local conditions (opposite) are verified directly for the Exit functor, thus since $\mathrm{st}_*\,\mathcal{F}$ are maps from Exit, we get that $\mathrm{st}_*\,\mathcal{F}$ is a cone-local sheaf. Next, since $X \times \mathbb{R} \to X$ induces an equivalence on Exit by definition of $\mathcal{S}\mathrm{trat}$, we get that $\mathrm{st}_*\,\mathcal{F}$ is also constructible.

For the other direction, we need to show that the unit $\mathcal{F} \to \mathrm{st}_*\,\mathrm{st}^*\,\mathcal{F}$ is an equivalence for a cone-local constructible sheaf. Since both are sheaves, we reduce to basics $\mathbb{R}^i \times \mathrm{C}(Z)$, and since both are constructible we reduce to $\mathrm{C}(Z)$, and lastly by homotopy equivalence to $\overline{\mathrm{C}}(Z)$, where Z is a compact stratified space. Now we induct downwards on the maximal p s.t. $Z \cong \overline{\mathrm{C}}^p(L)$. The case $p < \dim Z$ being that of $Z = \Delta^{\dim Z}$, and the inductive step follows by a similar chain of arguments as the above, to show that we can write it as a longer cone. □

From this lemma, we see the following, for a presheaf $\mathcal{F} \in \mathrm{PShv}(\underline{\mathrm{Strat}})$:

- The *sheaf* condition implies that it is determined by its values on basics $\mathbb{R}^i \times \mathrm{C}(L)$.
- Further imposing the *constructible* condition means that it factors through $\mathcal{S}\mathrm{trat}$, i.e. determined by itss values on cones $\mathrm{C}(L)$.
- Further imposing the *cone-local* condition means that it is determined by its values on standard simplicies $\Delta^n = \overline{\mathrm{C}}^n(\emptyset)$.

Furthermore, we can impose the *consecutive* condition, that is for each n the following is sent to a pullback

$$\begin{array}{ccc} \Delta^{[1:1]} & \longrightarrow & \Delta^{[1:n]} \\ \downarrow & & \downarrow \\ \Delta^{[0:1]} & \longrightarrow & \Delta^{[0:n]} \end{array}$$

which evidently is equivalent to imposing the Segal condition on the other side.

Lastly, we can impose the *univalent* condition, that is the following is sent to a pullback

$$\begin{array}{ccc} \Delta^{\{0<2\}} \coprod \Delta^{\{1<3\}} & \longrightarrow & \Delta^{\{0<1<2<3\}} \\ \downarrow & & \downarrow \\ \Delta^0 \coprod \Delta^0 & \longrightarrow & \Delta^0 \end{array}$$

which evidently is equivalent to imposing the 2-out-of-6 property, i.e. complete condition on the other side.

Definition 20. $\mathcal{S}\mathrm{tri} \subset \mathrm{PShv}\left(\underline{\mathrm{Strat}}\right)$ is the full subcategory of univalent consecutive cone-local constructible sheaves.

Therefore we arrive at the following:

Theorem 21. *The adjunction further restricts to an equivalence* $\mathcal{S}\mathrm{tri} \cong \mathrm{Cat}_\infty = \mathrm{PShv}^{\mathrm{Segal,cplt}}(\Delta)$*, given by* $\mathcal{C} \in \mathrm{Cat}_\infty$ *mapped to* $X \mapsto \mathrm{Map}\left(\mathrm{Exit}\left(X\right), \mathcal{C}\right)$*.*