Nonabelian Chabauty Seminar – Neutral Tannakian Categories

Shay Ben Moshe

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1 Introduction

Our goal is to define the *fundamental group* in an algebraic situation. A definition using paths, homotopies and so on, will be problematic, as algebraic objects are very rigid, as opposed to topological objects. Therefore, we need another approach. The idea is to define something geometric in the topological situation, from which we can reconstruct the (usual) fundamental group, and then mimic the geometric construction in algebraic geometry and then "reconstruct" the fundamental group the same way. Specifically, denoting $G = \pi_1(X, x)$ (usual fundamental group), $G\mathrm{Set}$ and $\mathrm{Rep}_k(G)$ have geometric interpretations, but we will get to this later.

We will first review the reconstruction part, and once we are done with this we will move on to the definition of the fundamental group.

2 Baby Case

Let G be a (discrete) group. Look at the category of G-sets, $G\mathrm{Set}$. There is a forgetful functor $\omega : G\mathrm{Set} \to \mathrm{Set}$.

There is a homomorphism $G \to \mathrm{Aut}(\omega)$ given by $g \mapsto \eta_g$ where $\eta_g : \omega \Rightarrow \omega$ is the natural transformation given by $\eta_{g,X}(x) = g.x$, and since functions are equivariant this is clearly a natural transformation. We wish to show that this homomorphism is actually an isomorphism.

First of all, we note that ω is corepresentable, namely $\omega(X) \cong \hom_{G\mathrm{Set}}(G, X)$, where the natural isomorphism is given by $x \mapsto (g \mapsto g.x)$. The result now follows by the (co-)Yoneda lemma for GSet:

$$\begin{aligned}\mathrm{Aut}(\omega) &\overset{\text{for now just End}}{=} \mathrm{Nat}(\omega,\omega)\\ &= \mathrm{Nat}\left(\hom_{G\mathrm{Set}}(G,-),\hom_{G\mathrm{Set}}(G,-)\right)\\ &= \hom_{G\mathrm{Set}}(G,G)\\ &= G\end{aligned}$$

This means that the data of $\omega : G\mathrm{Set} \to \mathrm{Set}$ is enough to recover G.

3 Hints Towards A Generalization

We will later have to deal with a more general case, so let's try and extract as much as we can from the above.

First of all, let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category, denote by $k = \mathrm{End}(1)$ the group of endomorphisms of the unit. By an Eckmann-Hilton argument, k is actually an abelian group. Furthermore, it acts on every object naturally, that is each $a \in k$ gives a natural transformation $a : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$: given $1 \overset{a}{\to} 1$, and an object $X \in \mathcal{C}$ we have the map $X \overset{\sim}{\longrightarrow} X \otimes 1 \xrightarrow{\mathrm{id} \otimes a} X \otimes 1 \overset{\sim}{\longrightarrow} X$.

Let us return to our case. The categories GSet and Set are <u>closed</u> <u>symmetric monoidal</u>. The SM structure is the product, and the unit is the trivial G-set $*$ (and $*$ for Set). There is internal hom $[X, Y]$ which has the underlying set $\hom_{G\mathrm{Set}}(X, Y)$ with the action given by $(g.f)(x) = g.\left(f\left(g^{-1}.x\right)\right)$ (and just $\hom(X, Y)$ for Set).

Furthermore, in our case, $k = \mathrm{End}(1) = *$ in both categories, so we have a trivial action on both categories. It is also natural to identify Set with kSet. We note that $\omega : G\mathrm{Set} \to k\mathrm{Set}$ satisfies the following: it is <u>symmetric monoidal</u>, <u>faithful</u>, <u>preserves finite limits and colimits</u>, and it <u>commutes with the k-action</u>.

Note that the automorphisms $\mathrm{Aut}(\omega)$ were all symmetric monoidal, that is $\eta_{X \times Y} = \eta_X \times \eta_Y$ and $\eta_* = \mathrm{id}_*$. And indeed it was natural to ask only for symmetric monoidal automorphisms, $\mathrm{Aut}^{\otimes}(\omega)$.

We will later see that this was an $\mathbb{F}_1$ shadow of a statement that works over other fields.

4 Affine Group Schemes and Hopf Algebras

We will take a short interlude to recall some things about group schemes. Let k be some field.

Definition 1. A *group scheme* G is a group object in the category of schemes Sch_k. An *affine group scheme* is a group scheme whose underlying scheme is affine.

If we have time: Let's recall two ways to think about this definition. The easiest way is to consider the functor of points $G : \mathrm{CAlg}_k \to \mathrm{Set}$, the data of a group scheme is precisely a lift to a functor $G : \mathrm{CAlg} \to \mathrm{Grp}$. The other way is as follows, assume that $G = \operatorname{Spec} A$, i.e. it is affine. We have a multiplication $G \times G \to G$, identity $\operatorname{Spec} k = * \to G$ and an inverse $G \to G$. Since it is affine, this is the same data as a comultiplication $\Delta : A \to A \otimes A$, coidentity $\varepsilon : A \to k$, and coinverse $S : A \to A$, such that it is coassociative

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta \otimes \mathrm{id}} \\ A \otimes A & \xrightarrow{\mathrm{id} \otimes \Delta} & A \otimes A \otimes A \end{array}$$

counital $A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id} \otimes \varepsilon} A \otimes k \xrightarrow{\sim} A = A \xrightarrow{\mathrm{id}} A$, and has the coinverse $A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id} \otimes S} A \otimes A \to A = A \xrightarrow{\varepsilon} k \to A$. This exactly equips the k-algebra A with the structure of an (associative commutative coassociative) *Hopf algebra*.

Definition 2. An affine group scheme $G = \operatorname{Spec} A$ is *algebraic* if A is a finitely generated k-algebra. G is *pro-algebraic* if A is a colimit of finitely generated k-algebras.

5 A Generalization

Let k be a field, and let G be an affine group scheme over k. The category of finite-dimensional G-representations over k is denote by $\mathrm{Rep}_k(G)$. A description of the objects in the category is pairs (V, ρ), where $V \in \mathrm{Vect}_k$ is a finite-dimensional k-vector space, together with an action of G, i.e. a morphism of group schemes $\rho : G \to \mathrm{GL}_V$.

Again, we have a forgetful functor $\omega : \mathrm{Rep}_k(G) \to \mathrm{Vect}_k$, which just remembers the underlying k-vector space.

We note that here again, $\mathrm{Rep}_k(G)$ and Vect_k are closed symmetric monoidal. The SM structure is the tensor product, and unit is the trivial G-representation k. Unlike last time, the categories are abelian, so the hom's are not merely sets but abelian groups. This gives $k = \mathrm{End}(1)$ also the additive structure, which recovers the field. Further, the categories are equipped with a $k = \mathrm{End}(1)$-action, which means that the hom's are in fact k-vector space. One can also define the G-action, which shows that it is indeed closed. Moreover, every object is dualizable, and the dual is given by $(-)^* = [-, 1]$.

Now, our functor ω is symmetric monoidal, faithful, preserves finite limits and colimits, which in this context is usually called exact, and commutes with the k-action, which in this context is usually called k-linear.

This is very similar to the case $G\text{Set} \to k\text{Set}$ from before, so there is a hope for recovering G from $\omega : \text{Rep}_k(G) \to \text{Vect}_k$. For this we wish to construct a k-group scheme $\underline{\text{Aut}}^{\otimes}(\omega)$.

If we have time: The corresponding usual categorical notion is the automorphism group of SM natural transformations $\text{Aut}^{\otimes}(\omega)$. Concretely, for each $(V,\rho) \in \text{Rep}_k(G)$ we need $\eta_{V,\rho} : V \to V$ such that, first it is a natural transformation of ω, i.e. for $T : (V,\rho) \to (W,\tau)$ we have:

$$\begin{array}{ccc} V & \xrightarrow{\omega(T)} & W \\ \downarrow{\scriptstyle \eta_{V,\rho}} & & \downarrow{\scriptstyle \eta_{W,\tau}} \\ V & \xrightarrow{\omega(T)} & W \end{array}$$

and second, it is SM, that is $\eta_{(V,\rho)\otimes(W,\tau)} = \eta_{(V,\rho)} \otimes \eta_{(W,\tau)}$and $\eta_k = \text{id}_k$. Such element $\eta \in \text{Aut}^{\otimes}(\omega)$ will give an element in $\underline{\text{Aut}}^{\otimes}(\omega)(R)$ for each k-algebra R as follows. The map $\eta_{V,\rho} : V \to V$ gives a map $\eta_{V,\rho} \otimes \text{id}_R : V \otimes_k R \to V \otimes_k R$, and it commutes with every map $T \otimes \text{id}_R$ coming from $T : (V,\rho) \to (W,\tau)$ as above.

We turn this into the definition of $\underline{\text{Aut}}^{\otimes}(\omega)(R)$, namely, it is the group of $\{\eta_{V,\rho} : V \otimes_k R \to V \otimes_k R\}_{V,\rho}$, that commute with every $T \otimes \text{id}_R$. Clearly, for every $g \in G(R)$, we get such a map by $\eta_{V,\rho} = \rho(g)$. This gives a morphism of group schemes $G \to \underline{\text{Aut}}^{\otimes}(\omega)$.

Theorem 3. *The morphism $G \to \underline{\text{Aut}}^{\otimes}(\omega)$ is an isomorphism of group schemes.*

6 Neutral Tannakian Categories

As we have seen, given the category $\text{Rep}_k(G)$ and the functor $\omega : \text{Rep}_k(G) \to \text{Vect}_k$, we can recover G. Now, assuming we have some category $\mathcal{C}$, we wish to "recover G" from it. That is, we wish to find a group scheme G and admit $\mathcal{C}$ as $\text{Rep}_k(G)$. For this to happen, we certainly need $\mathcal{C}$ to have all the properties and structure that $\text{Rep}_k(G)$ has, and all of the properties and structure described before are in fact sufficient, that is we have the following theorem:

Theorem 4 (Tannakian Reconstruction)**.** *Let $(\mathcal{C}, 1, \otimes)$ be a closed symmetric monoidal abelian category such that every object is dualizable. Denote $k = \text{End}(1)$. Let $\omega : \mathcal{C} \to \text{Vect}_k$ be a symmetric monoidal faithful exact k-linear functor. Then*

1. *there is an affine group scheme $G = \underline{\text{Aut}}^{\otimes}(\omega)$,*
2. *ω factors to an equivalence $\mathcal{C} \to \text{Rep}_k(G)$.*

Definition 5. A category satisfying the conditions above, and that admits ω satisfying the conditions above, is called a *neutral Tannakian category.* Any such ω is called a *fiber functor.*

We don't have time to prove this theorem, but let's give a sketch. First, we note that for an affine group scheme $G = \operatorname{Spec} B$, a G-representation is the same data as a B-comodule, i.e. $\operatorname{Rep}_k(G) = \operatorname{Comod}_B$, therefore the theorem can be described in these terms, and we need to produce B and the corresponding equivalence $\mathcal{C} \to \operatorname{Comod}_B$.

The first part is to construct a k-coalgebra. To do this, we don't use the symmetric monoidal structure. For each $X \in \mathcal{C}$ we consider the full subcategory of subquotients $\langle X \rangle \subseteq \mathcal{C}$, and the restriction $\omega\mid_{\langle X \rangle} : \langle X \rangle \to \operatorname{Vect}_k$. The k-algebra $A_X = \operatorname{End}\left(\omega\mid_{\langle X \rangle}\right)$ acts on $\omega\mid_{\langle X \rangle}(Y)$ for $Y \in \langle X \rangle$, and so the functor actually factors as $\omega\mid_{\langle X \rangle} : \langle X \rangle \to \operatorname{Mod}_{A_X} = \operatorname{Comod}_{B_X}$ where $B_X = A_X^*$. We then define $B = \operatorname{colim} B_X$, and the functor $\omega : \mathcal{C} \to \operatorname{Vect}_k$ factors as an equivalence $\omega : \mathcal{C} \to \operatorname{Comod}_B$.

For the second part, we assume that $\mathcal{C}$ is symmetric monoidal, and the functor is symmetric monoidal. Then, via the equivalence $\omega : \mathcal{C} \to \operatorname{Comod}_B$, we get a symmetric monoidal structure on Comod_B, which gives B the structure of an associative commutative k-algebra. Therefore, we can define $G = \operatorname{Spec} B$, and the coalgebra structure gives it the structure of a monoid scheme, and $\mathcal{C} \to \operatorname{Rep}_k(G)$ is an equivalence. Taking similar notations to before, we get that $G = \underline{\operatorname{End}}^{\otimes}(\omega)$.

The last part is to obtain the inverse $S : B \to B$, that is to give G the structure of a group scheme rather then a monoid scheme. We assumed that all objects in $\mathcal{C}$ are dualizable. If $\eta : F \Rightarrow G$ is a SM natural transformation between SM functors between SM categories with all objects dualizable, we can define an inverse by $G(X) \xrightarrow{\sim} G(X^*)^* \xrightarrow{(\eta_{X^*})^*} F(X^*)^* \xrightarrow{\sim} F(X)$, so it is automatically a natural equivalence. That is, in this case $\underline{\operatorname{End}}^{\otimes}(\omega) = \underline{\operatorname{Aut}}^{\otimes}(\omega)$, and we are done.

We remark that B was the colimit of B_X, which in turn are finitely generated, so G is pro-algebraic.

7 The Fundamental Group

Let X be a connected topological space, $x_0 \in X$, denote $G = \pi_1(X, x_0)$. We denote by $\tilde{X}$ the universal cover, and recall that it has a free transitive fiberwise G-action. The category $G\mathrm{Set}$ is known to be equivalent to the category of covers of X. Although this is valid approach to try to mimic in the algebraic situation, and using the first part of the lecture we can actually define the *etale fundamental group,* this is not the path we will follow. Rather, we will use the category $\operatorname{Rep}_k(G)$ of k-representations. We claim that this category is equivalent to the category of *k-vector bundles with flat connection.* For the first direction, let V be a G-representation, $\tilde{X} \times V$ is a vector bundle $\tilde{X}$ with flat connection. It is equipped with a G-action, and the quotient $\left(\tilde{X} \times V\right)/G$ is a vector bundle over X, with an induced flat connection. For the other direction, let $E \to X$ be a vector bundle with a flat connection. Given a path γ from x_0 to itself, the connection gives a map $E_{x_0} \to E_{x_0}$, and the flatness of the connection guarantees that it depends only the homotopy class, so we get a map $G \to \mathrm{GL}_{E_{x_0}}$, i.e. a G-representation. Therefore we can define the composition $\omega : \mathrm{Conn}(X) \cong \mathrm{Rep}_k(G) \to \mathrm{Vect}_k$, which takes a bundle to the fiber over x_0. The category is indeed neutral Tannakian, and using the theorem we can recover the pro-algebraic approximation $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ of $\pi_1(X, x_0)^{\mathrm{alg}} = G$.

Luckily, in Shaul's talk we saw how to define the category of vector bundles with flat connection in the algebraic situation. Namely, let k be a field of characteristic 0, and let X be a *geometrically connected smooth scheme* over k. We have a definition of the category of flat connections $\mathrm{Conn}(X)$. Now, let $x_0 = \mathrm{Spec}\, k(x_0) \to X$ be a point of X, then we have a functor which takes the fiber of E at x_0, i.e. $\omega : \mathrm{Conn}(X) \to \mathrm{Vect}_{k(x_0)}$ given by $E \mapsto E_{x_0}$. The categories are indeed closed symmetric monoidal and abelian, and every object is dualizable. By the fact that X is geometrically connected, $\mathrm{End}(1) = k(x_0)$. It can be checked that ω is indeed a fiber functor. Therefore we can "reconstruct" a group which is defined to be the *(pro-algebraic) fundamental group* $\pi_1(X, x_0) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Furthermore, suppose $\mathcal{C}$ is a full subcategory of $\mathrm{Conn}(X)$ which is neutral Tannakian (namely, abelian subcategory, closed under tensor product, contains the unit and closed to taking duals), then we can take the composition $\mathcal{C} \to \mathrm{Conn}(X) \to \mathrm{Vect}_{k(x_0)}$ and reconstruct to get $\pi_1(X, x_0; \mathcal{C})$.

8 The Unipotent Fundamental Group

Lastly, it is a familiar fact e.g. from the theory of Lie algebras that it is a good idea to look at nilpotent Lie algebras. The analogue of a nilpotent in a Lie algebra in a group is a *unipotent*. In GL_n this can be defined as an element A such that $A - I \in \mathrm{Mat}_n$ is nilpotent, or equivalently, all eigenvalues of A are 1. It turns out that every algebraic group is linear (i.e. embeds in GL_n), and this gives an invariant notion of unipotent elements. So it seems like a good idea to study the maximal unipotent quotient group $\pi_1^{\mathrm{uni}}(X, x_0)$. Luckily for us, this also fits nicely to the framework of Tannakian reconstruction, namely there is a full subcategory $\mathrm{Conn}^{\mathrm{uni}}(X) \subseteq \mathrm{Conn}(X)$ that can be described geometrically whose reconstruction gives $\pi_1^{\mathrm{uni}}(X, x_0)$, which we now wish to describe.

Going back to the topological case, we recall that an object in the category was a representation $G \to \mathrm{GL}_{E_{x_0}}$, being unipotent means giving a *unipotent representation*, i.e. acting via unipotent matrices. A unipotent matrix is conjugate to an upper triangular matrix with diagonal 1. Such a matrix determines a series of invariant subspaces $0 = V_0 < V_1 < \cdots < V_{n-1} < V_n = E_{x_0}$ with $\dim V_i = i$, that is a *complete flag*, where V_i is spanned by the first i elements in the triangulating basis. Furthermore, the diagonal begin 1 is equivalent to the action on the associated graded V_i / V_{i-1} being trivial. It turns out if a group acts on a vector space via unipotent elements (i.e. we have a unipotent representation), then we can conjugate them simultaneously to the above form. This means that a unipotent representation is built as an iterated extension of trivial representations. Going through the equivalence $\mathrm{Conn}(X) \cong \mathrm{Rep}_k(G)$, this corresponds to bundles which are iterated extensions of the trivial bundle with the trivial connection, so we can define the full subcategory $\mathrm{Conn}^{\mathrm{uni}}(X)$ of such bundles.

This is easily translated to the algebraic situation which gives the desired subcategory $\mathrm{Conn}^{\mathrm{uni}}(X) \subseteq \mathrm{Conn}(X)$, whose reconstruction gives the *unipotent (pro-algebraic) fundamental group* $\pi_1^{\mathrm{uni}}(X, x_0) = \pi_1\left(X, x_0; \mathrm{Conn}^{\mathrm{uni}}(X)\right)$. Similarly, one can also define *quasi-unipotent* to be element s.t. some power is unipotent, and get the group $\pi_1^{\mathrm{quni}}(X, x_0)$.