Prismatic Cohomology – Prisms

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The goal of this talk is to define *prisms*, which are some of the main ingredients in this theory, and to prove some of their basic properties. A few reminders from previous lectures:

Definition 1. A δ*-ring* is a ring A equipped with a function $\delta : A \to A$, with axioms that make the map $\phi_\delta(x) = \phi(x) = x^p + p\delta(x)$ into a ring homomorphism. An element $x \in A$ is called *distinguished* if $\delta(x)$ is a unit (somewhat similar to a uniformizer).

1 Definitions

Without further ado, we move towards the definition of a prism.

Definition 2 (The category of δ*-pairs*)**.** A δ*-pair* is (A, I) where A is a δ-ring (the δ-function is omitted from the notation) and $I \lhd A$ is any ideal. A morphism $(A, I) \to (B, J)$ is a morphism of δ-rings $A \to B$ that carries I into J.

Definition 3 (The category of *prisms*)**.** A *prism* is a δ-pair (A, I) such that:

1. I defines a Cartier divisor (locally principal, generated by a non-zero-divisor),
2. A is derived (p, I)-complete (in particular p, I are in $\operatorname{rad}(A)$),
3. $p \in I + \phi(I) A$.

The category of prisms is the full subcategory of δ-pairs on the prisms.

The prism (A, I) is called:

1. *perfect* if A is a perfect δ-ring, i.e. $\phi : A \to A$ is an isomorphism,
2. *bounded* if A/I has bounded p^∞-torsion (i.e. $A[p^\infty] = A[p^m]$ for m large enough),
3. *orientable* if I is principal, a choice of a generator is called an *orientation*,

4. *crystalline* if $I = (p)$, in particular bounded and orientable.

Example 4 (Crystalline)**.** For A a p-torsion-free and p-complete δ-ring, e.g. $\mathbb{Z}_p$, the pair $(A, (p))$ is a crystalline prism, and any crystalline prism is of this form.

Example 5 (q-de Rham)**.** $A = \mathbb{Z}_p\left[[q-1]\right]$ with δ-structure $\phi(q) = q^p$, and $I = \left([p]_q\right)$ is a bounded orientable prism.

Example 6 (Universal oriented)**.** Let $A_0 = \mathbb{Z}_{(p)}\left\{d, \delta(d)^{-1}\right\}$ be the localization of the free δ-ring on d, and denote $A = (A_0)^{\wedge}_{p,d}$. Then $(A, (d))$ is a bounded oriented prism. In fact, it is the universal oriented prism.

2 Properties & More

The following lemma will mostly be applied for prisms:

Lemma 7. *Let (A, I) be a δ-pair such that I is locally principal, and p, I are in* $\operatorname{rad}(A)$*, then TFAE:*

1. $p \in I^p + \phi(I) A$,
2. $p \in I + \phi(I) A$,
3. *there exists a faithfully flat map of δ-rings $A \xrightarrow{\text{ff}} A'$ that is an ind-Zariski localization such that $IA' = (d)$ with d distinguished and $d, p \in \operatorname{rad}(A')$.*

Proof. (1) $\implies$ (2) is trivial.

(2) $\implies$ (3): Let $(g_1, \dots, g_n) = A$ such that $IA[1/g_i]$ is principal. Denote $B = \prod A[1/g_i]$, then $A \to B$ is faithfully flat and $IB = (d)$ is principal. Finally define A' to be the localization along $V(p, d)$ of B, so that $d, p \in \operatorname{rad}(A')$. $A \to B \to A'$ is still faithfully flat (flatness is immediate, faithfulness is since $d, p \in \operatorname{rad}(A)$). By assumption $p \in \operatorname{rad}(A)$, so any localization of A admits a unique compatible δ-A-algebra structure (ϕ sends units to units, hence $S^{-1}A$ and $\left\{S, \phi(S), \phi^2(S), \dots\right\}^{-1} A$ define the same localization, and ϕ on the latter is a lift of Frobenius). A' is a finite product of such so it has a unique compatible δ-A-algebra structure. By construction $IA' = (d)$, and by assumption we have $p \in (d, \phi(d))$, therefore by a previous lemma about δ-rings, d is distinguished.

(3) $\implies$ (1): We can check the condition after faithfully flat base change, thus enough to check that $p \in (d^p, \phi(d)) \subseteq A'$. Since d is distinguished, $\delta(d)$ is a unit, and $\phi(d) = d^p + p\delta(d)$, so the condition is satisfied. □

Lemma 8 (Rigidity)**.** *Let $(A, I) \to (B, J)$ be a map of prisms. Then $I \otimes_A B \xrightarrow{\sim} J$, and in particular $J = IB$.*

Further, if $A \to B$ is a map of δ-rings with B derived (p, I)-complete, then (B, IB) is a prism iff $B[I] = 0$.

Proof. The idea is to work locally where the ideals are generated by a distinguished element and use their irreducibility. Specifically, using the previous lemma we choose $A \xrightarrow{\text{ff}} A'$ such that $IA' = (d)$ with d distinguished and $d, p \in \text{rad}(A')$. We want to choose similarly B' with $JB' = (e)$ etc., such that $B \to B'$ is faithfully flat and, and we have a map $A' \to B'$ making the square commute. For this take $A' \otimes_A B$ localized at $V(p, J)$ (which is ff over B), and then apply the previous lemma for it.

$$\begin{array}{ccc} A & \xrightarrow{\text{ff}} & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\text{ff}} & B' \end{array}$$

By assumption I is sent to J in B, thus $(d) = IB' \subseteq JB' = (e)$, so that $d = ef$ for some $f \in B'$. Recall that d is distinguished and $p, e \in \text{rad}(B')$, so we get that f is a unit, so $(d) = (e)$, i.e. $IB' = JB'$. The result follows by faithfully flat descent.

For the second statement, note that $B[I] = 0$ iff $I \otimes_A B \xrightarrow{\sim} IB$. From the first part, if (B, IB) is a prism then $B[I] = 0$. If $B[I] = 0$, then IB is an invertible B-module, so a Cartier divisor. Also, $p \in IB + \phi(I) B$ follows from the condition on A. Lastly, B is derived (p, I)-complete by assumption, so (B, IB) is a prism. □

Lemma 9. *Let (A, I) be a prism. Then $\phi(I) A$ is principal and any generator is distinguished. Moreover, the invertible A-modules $\phi^*(I) = I \otimes_{A,\phi} A$ and I^p are trivial (isomorphic to A).*

Proof. It is enough to find one distinguished generator for $\phi(I) A$, since another generator differs by a unit i.e. also distinguished. By lemma 7, $p \in I^p + \phi(I) A$, write $p = a + b$. We claim that $(b) = \phi(I) A$ and that b is distinguished. Again using lemma 7 choose $A \xrightarrow{\text{ff}} A'$ with $IA' = (d)$. As $IA' = (d)$, we have $a = xd^p$ and $b = y\phi(d)$ for $x, y \in A'$. Enough to show that y is a unit, since then $\phi(I) A'$ is generated by b, and it is distinguished (because ϕ and multiplication by units send distinguished to distinguished). Since $p, d \in \text{rad}(A')$, enough to show that y is a unit in $A'/(p, d)$, i.e. $A'/(p, d, y) = 0$. Assume not, then by localizing we can assume that $y \in \text{rad}(A)$. Since $p = a + b = xd^p + y(d^p + p\delta(d))$ we get $p(1 - y\delta(d)) = d(d^{p-1}(x + y))$. Since p is distinguished and $y \in \text{rad}(A')$, the LHS is distinguished. Since d is also distinguished, we get that $d^{p-1}(x + y)$ is a unit, thus d is a unit which contradicts $d \in \text{rad}(A')$, and we are done.

We won't prove the second part, the idea is that over A/p, $\phi^*(I) \cong I^p$, and $p \in \text{rad}(A)$ so they identify in A, thus it suffices to check for $\phi^*(I)$. This is checked on closed points, by passing to $(A_{\text{perf}})^{\wedge}_p$. □

Lemma 10 (Properties of bounded prisms)**.** *Let (A, I) be a bounded prism (i.e. A/I has bounded p^∞-torsion), then:*

1. *A is classically (p, I)-complete.*

2. *Let $M \in \mathcal{D}(A)$ be a (p, I)-completely flat A-complex (not in the paper, M derived (p, I)-complete). Then M is discrete and classically (p, I)-complete. For any $n \geq 0$, we have $M[I^n] = 0$ and M/I^n has bounded p^∞-torsion.*

3. *The functor $B \mapsto (B, IB)$ induces an equivalence from the category of (p, I)-completely (faithfully) flat δ-A-algebras B (not in the paper, B derived (p, I)-complete), to the category of (faithfully) flat maps $(A, I) \to (B, J)$ of prisms.*

4. *(locally orientable) There exists a (p, I)-completely faithfully flat map of δ-ring $A \to B$, s.t. $IB = (d)$ for $d \in B$ distinguished non-zero-divisor, which can be chosen to be the derived (p, I)-completion of an ind-Zariski localization of A. In particular, $(A, I) \to (B, (d))$ is a faithfully flat map of bounded prisms.*

Proof. For (1), we have

$$\begin{aligned} A &= \operatorname*{Rlim}_n \operatorname*{Rlim}_m A//(I^n, p^m) \\ &= \operatorname*{Rlim}_n \operatorname*{Rlim}_m (A/I^n)//(p^m) \\ &= \operatorname*{Rlim}_n \operatorname*{Rlim}_m A/(I^n, p^m) \\ &= \lim_n \lim_m A/(I^n, p^m) \end{aligned}$$

The first step is using the derived (p, I)-completeness, the second is because I^n is locally generated by non-zero-divisors, the third is because A/I, thus also A/I^n, has bounded p^∞-torsion, and the last since the limits are direct and maps are surjective (Mittag-Leffler).

The argument for (2) is similar to (1). We first work separately to get the properties for $M//I^n$ using its p-completely flatness, and in particular to get that it is discrete (thus $M[I^n] = 0$). Then, taking $n \to \infty$ and using the derived I-completeness we get that M itself is discrete. Arguing as above we get that it is also classically (p, I)-complete.

For (3), the inverse is of course given by $(B, IB) \mapsto B$ (by rigidity), we check that the functors land where they should. If B is a (p, I)-completely (faithfully) flat δ-A-algebra, then by rigidity it is enough to check that $B[I] = 0$ which follows from (2). For the other direction, if $(A, I) \to (B, J)$ is a (faithfully) flat map of prisms, since by rigidiy $J = IB$, $A \to B$ is in particular (p, I)-completely (faithfully) flat.

For (4), take B to the the derived (p, I)-completion of A' from lemma 7. By (2) B is discrete, B/IB has bounded p^∞-torsion, and $B[I] = 0$, so by rigidity (B, IB) is a prism. ∎

2.1 Perfect Prisms

Lemma 11 (Properties of perfect prisms). *Let (A, I) be a perfect prism (ϕ is an isomorphism), then:*

1. *I is principal and any generator is a distinguished element.*
2. *(A, I) is bounded, and in particular A is classically (p, I)-complete (by the previous lemma).*

Proof. We have seen that $\phi(I)A$ is principal and any generator is distinguished, and by assumption ϕ is an isomorphism so the same is true for I. For (2), in general for δ-rings, if ϕ is injective then A is p-torsion-free. Choose a distinguished generator d, we have seen that perfectness together with p-torsion-freeness imply that A/d has bounded p^∞-torsion. □

Lemma 12 (Perfection). *Let (A, I) be a prism. Denote by $A_{\text{perf}} = \operatorname{colim}_\phi A$ the perfection. Then $IA_{\text{perf}} = (d)$, where d is distinguished, p, d are non-zero-divisors, and $A_{\text{perf}}/d[p^\infty] = A_{\text{perf}}/d[p]$. In particular the derived (p, I)-completion of A_{perf}, denoted A_∞, agrees with the classical completion, and $(A, I)_{\text{perf}} = (A_\infty, IA_\infty)$ is the universal perfect prism under (A, I).*

Proof. As $A \to A_{\text{perf}}$ factors through ϕ, and $\phi(I)A$ is generated by a distinguished element, so does $IA_{\text{perf}} = (d)$. The other two properties follow from properties of the δ-ring structure. By p-torsion-freeness we have that the derived and classical p-completions coincide to give $(A_{\text{perf}})^\wedge_p$. Here d is still non-zero-divisor, also the quotient by d^n is still derived and classical p-complete. Therefore we get that the derived and classical d-completion of $(A_{\text{perf}})^\wedge_p$ coincide, and give $A_\infty = (A_{\text{perf}})^\wedge_{(p,d)}$. It is now clear that it is a prism, and the universal property is also obvious. □

Remark 13. We don't have the time to explain (or define) this, but it is worth remarking that the functor $(A, I) \mapsto A/I$ gives an equivalence of categories from perfect prisms to perfectoid rings (with inverse $R \mapsto (A_{\text{inf}}(R), \ker\theta)$).

2.2 Site of Prisms

Theorem 14. *The opposite category of bounded prisms (A, I), with topology where covers are faithfully flat maps of prisms, forms a site.*

Proof. There are three axioms to check. First, isomorphisms are covers, which is clear. Second, composition of covers is a cover, is also clear. Lastly, we need to check that the pushout of a cover along an arbitrary map is a cover. Assume we have prisms (using rigidity to determine the ideals)

$$\begin{array}{ccc} (A, IA) & \xrightarrow{b,\mathrm{ff}} & (B, IB) \\ \downarrow{\scriptstyle c} & & \vdots \\ (C, IC) & \xrightarrow{d,\mathrm{ff}} & (D, ID) \end{array}$$

and we want to construct the pushout such that d is also faithfully flat. Take $D = \left(B \otimes_A^L C\right)^{\wedge}_{(p,I)}$ the derived (p, I)-completion of $B \otimes_A^L C$. By standard properties, D is (p, I)-completely faithfully flat over C. By properties of bounded prisms it follows that D is discrete, and that $(C, IC) \to (D, ID)$ is a faithfully flat map of bounded prisms. One checks that it serves as a pushout of b along c. □

Theorem 15. *The functor that carries (A, I) to A (resp. A/I) is a sheaf for this topology, with vanishing higher cohomology on any (A, I).*

Proof. We denote by $\underline{(-)}$ the functor $\underline{(A, I)} = A$. Let $(A, I) \to (B, IB)$ be faithfully flat map (cover). Denote by $N^\bullet$ the Cech nerve of (B, IB), and we need to check that $\underline{(A, I)} = A \xrightarrow{\sim} \lim_\Delta \underline{N^\bullet}$ and that higher Rlim vanish. N^k is by definition the pushout of (B, IB) with itself along (A, I) $k + 1$-times, so as we have seen in the previous proof, $\underline{N^k} = \left(B^{\otimes_A^L k+1}\right)^{\wedge}_{(p,I)}$, i.e. the derived completion of the derived Cech nerve of $A \to B$. Thus we get:

$$\begin{aligned}
\underset{\Delta}{\mathrm{Rlim}}\, \underline{N^k} &= \underset{\Delta}{\mathrm{Rlim}} \left(B^{\otimes_A^L k+1}\right)^{\wedge}_{(p,I)} \\
&= \underset{\Delta}{\mathrm{Rlim}}\, \underset{n}{\mathrm{Rlim}} \left(B^{\otimes_A^L k+1}\right) // \left(p^n, I^n\right) \\
&= \underset{n}{\mathrm{Rlim}}\, \underset{\Delta}{\mathrm{Rlim}} \left(B^{\otimes_A^L k+1}\right) // \left(p^n, I^n\right) \\
&\overset{\star}{=} \underset{n}{\mathrm{Rlim}}\, A // \left(p^n, I^n\right) \\
&= A^{\wedge}_{(p,I)} \\
&= A
\end{aligned}$$

where in $\star$ we use faithfully flat descent. The argument for A/I is similar, use the fact that $X/I \xrightarrow{\sim} X^{\wedge}_I/I$. □