Grothendieck-Lefschetz and the Weil Conjectures (except RH)

Shay Ben Moshe

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Tony Feng has really good notes on this topic in [Fen17]. Milne has a great set of note on Etale cohomology the Weil conjectures [Mil13], the proof of the last theorem state in these notes is in chapter 25 using chapter 23, and of the main theorem is in chapter 29. Also useful is the book [FK88], and especially chapters II.3 and II.4.

The goal of this talk is to explain the different versions of Grothendieck-Lefschetz trace formula, say something about its proof, and deduce the Weil Conjectures (except for the Riemann Hypothesis). In this area there a lot of confusing ideas and definitions, but some of them come from linear algebra while some of them come from algebraic geometry. Therefore, I will try to separate the different parts, so I will start by doing analogues over a point and over set, which involve no algebraic geometry, and only then move on to schemes. This is a somewhat non-standard presentation of the material, but hopefully, it will make it easier to digest.

1 Point

Let $\varphi\colon V \to V$ a linear operator on a finite dimensional vector space. We define its L-function as the unusually normalized characteristic polynomial

$$L(\varphi, t) := \det(1 - t\varphi)^{-1} = \frac{1}{(1 - a_1 t) \cdots (1 - a_n t)},$$

where a_i are the eigenvalues.

Lemma 1. $L(\varphi, t) = \exp\left(\sum_{r=1}^{\infty} \mathrm{Tr}(\varphi^r) \frac{t^r}{r}\right)$.

Proof. Recall the Taylor series of $-\log(1 - t) = \sum_{r=1}^{\infty} \frac{t^r}{r}$. Write φ as an upper-triangular matrix with the eigenvalues a_i on the diagonal, then we get

$$\begin{aligned}\log L(\varphi, t) &= \sum_i \left(-\log\left(1 - a_i t\right)\right) \\ &= \sum_i \left(\sum_{r=1}^{\infty} \frac{(a_i t)^r}{r}\right) \\ &= \sum_{r=1}^{\infty} \left(\sum_i a_i^r\right) \frac{t^r}{r} \\ &= \sum_{r=1}^{\infty} \operatorname{Tr}(\varphi^r) \frac{t^r}{r}\end{aligned}$$

□

Definition 2. Let $\mathcal{G}$ be a bounded chain complex of finite dimensional vector spaces, and let $\varphi\colon \mathcal{G} \to \mathcal{G}$ be an operator. We will use the shorter notation $\operatorname{Tr}(\varphi \mid \mathcal{G}) = \operatorname{Tr}(\varphi \mid \mathrm{H}^*\mathcal{G}) = \sum (-1)^i \operatorname{Tr}(\varphi \mid \mathrm{H}^i\mathcal{G})$ and $\det(\varphi \mid \mathcal{G}) = \det(\varphi \mid \mathrm{H}^*\mathcal{G}) = \prod \det(\varphi \mid \mathrm{H}^i\mathcal{G})^{(-1)^i}$ (in fact we will use the latter for $1 - t\varphi$ rather than φ).

Evidently, the previous lemma generalizes to this situation.

2 Set

Let be a finite set $\overline{X}$ with an endomorphism $F\colon \overline{X} \to \overline{X}$. We denote by $\overline{X}(F^r)$ the fixed points by F^r (in fact, we don't need $\overline{X}$ to be finite, but that each $\overline{X}(F^r)$ is finite, and that their union is $\overline{X}$). For each $x \in \overline{X}$ we define its *degree* d_x as the minimal number such that $F^{d_x}(x) = x$. We define the *closed points* as $\left|\overline{X}\right| := \overline{X}/F$ the orbits under the F action. Note that for each $\tilde{x} \in \left|\overline{X}\right|$ there are d_x lifts to $\overline{X}(F^r)$ if $d_x \mid r$ and 0 otherwise.

Definition 3. A *sheaf* on $\overline{X}$ (equipped with $F\colon \overline{X} \to \overline{X}$) is a family of finite dimensional vector spaces $V = (V_x)_{x \in \overline{X}}$ together with a map $\varphi\colon F^*V \to V$, i.e. a map $V_{F(x)} \to V_x$ for every $x \in \overline{X}$. In this was, for $x \in \overline{X}(F^r)$ we get a morphism $\varphi_x^r\colon V_x = V_{F^r(x)} \to V_{F^{r-1}(x)} \to \cdots \to V_x$. Also, for $\tilde{x}$, we denote by the horrible notation $\varphi_{\tilde{x}} := \varphi_x^{d_x}\colon V_x \to V_x$ for some lift of x.

Remark. The definition of $\varphi_{\tilde{x}}$ depends on the choice of a lift x, but only up to an isomorphism. In particular, its eigenvalues don't.

Definition 4. Let $f\colon \overline{X} \to \overline{Y}$ be a morphism of sets with an endomorphism. For a sheaf W on Y we define a sheaf f^*W on X by $f^*W_x = W_{f(x)}$, and similarly $f_!V_y = \bigoplus_{f(x)=y} V_x$.

Definition 5 (Sheaf to function correspondence)**.** Given $\overline{X}$ and a sheaf V, we define the function $T^rV\colon \overline{X}(F^r) \to \overline{\mathbb{Q}}$ by $T^rV(x) = \operatorname{Tr}(\varphi_x^r)$.

Definition 6. For a morphism $f\colon \overline{X} \to \overline{Y}$ we define

1. $f^*\colon \overline{\mathbb{Q}}^{\overline{Y}(F^r)} \to \overline{\mathbb{Q}}_\ell^{\overline{X}(F^r)}$ by $f^*T(x) = T(f(x))$.
2. $f_!\colon \overline{\mathbb{Q}}^{\overline{X}(F^r)} \to \overline{\mathbb{Q}}^{\overline{Y}(F^r)}$ by $f_!T(y) = \sum_{f(x)=y} T(x)$.

Proposition 7. *Let $f\colon \overline{X} \to \overline{Y}$ be a morphism, then $f^*T^r = T^r f^*$.*

Proof. Immediate. □

Proposition 8 (Yanovski trace formula)**.** *Let $f\colon \overline{X} \to \overline{Y}$ be a morphism, then $f_!T^r = T^r f_!$.*

Proof. This follows from the fact that trace sends direct sum to a sum. □

Example 9. The case $f\colon \overline{X} \to \overline{Y} = *$ and V constant 1-dimensional with $\varphi = \mathrm{id}$ is the obvious equality $\#\overline{X}(F^r) = \sum_{\overline{X}(F^r)} \mathrm{Tr}(\mathrm{id}) = \mathrm{Tr}\left(\bigoplus_{\overline{X}(F^r)} \mathrm{id}\right)$.

Definition 10. We define the L-function using the local L-functions,

$$L\left(\overline{X}, V, t\right) := \prod_{\tilde{x}\in\left|\overline{X}\right|} L\left(\varphi_{\tilde{x}}, t^{d_{\tilde{x}}}\right) = \prod_{\tilde{x}\in\left|\overline{X}\right|} \det\left(1 - t^{d_{\tilde{x}}}\varphi_{\tilde{x}}\right)^{-1}.$$

Proposition 11. $L\left(\overline{X}, V, t\right) = \exp\left(\sum_{r=1}^{\infty}\left(\sum_{x\in\overline{X}(F^r)} \mathrm{Tr}\left(\varphi_x^r\right)\right)\frac{t^r}{r}\right)$.

Proof. Indeed

$$\begin{aligned}
L\left(\overline{X}, V, t\right) &= \prod_{\tilde{x}\in\left|\overline{X}\right|} L\left(\varphi_{\tilde{x}}, t^{d_{\tilde{x}}}\right) \\
&= \prod_{\tilde{x}\in\left|\overline{X}\right|} \exp\left(\sum_{m=1}^{\infty} \mathrm{Tr}\left(\varphi_{\tilde{x}}^m\right)\frac{t^{d_{\tilde{x}}m}}{m}\right) \\
&= \exp\left(\sum_{m=1}^{\infty}\sum_{\tilde{x}\in\left|\overline{X}\right|} \mathrm{Tr}\left(\varphi_{\tilde{x}}^m\right) d_{\tilde{x}}\frac{t^{d_{\tilde{x}}m}}{d_{\tilde{x}}m}\right) \\
&= \exp\left(\sum_{r=1}^{\infty}\sum_{\tilde{x}\in\left|\overline{X}\right|, d_{\tilde{x}}|r} \mathrm{Tr}\left(\varphi_{\tilde{x}}^{r/d_{\tilde{x}}}\right) d_{\tilde{x}}\frac{t^r}{r}\right) \\
&= \exp\left(\sum_{r=1}^{\infty}\left(\sum_{x\in\overline{X}(F^r)} \mathrm{Tr}\left(\varphi_x^r\right)\right)\frac{t^r}{r}\right)
\end{aligned}$$

□

Corollary 12 (Yanovski trace formula for L-functions)**.** *Let $f\colon \overline{X} \to \overline{Y}$ be a morphism and V a sheaf on $\overline{X}$, then $L\left(\overline{X}, V, t\right) = L\left(\overline{Y}, f_! V, t\right)$. In particular, for $\overline{Y} = *$ we get $L\left(\overline{X}, V, t\right) = \det\left(1 - t\varphi \mid f_! V\right)^{-1}$.*

Remark 13. The Yanovski trace formula is a local-to-global kind of a thing, we compute something defined locally (the L-function, as the product of the local L-functions) via something defined globally (the "cohomology" $f_! V$).

Once more, the whole thing generalizes if we replace vector spaces V by chain complexes $\mathcal{G}$.

3 Scheme

Let $X/\mathbb{F}_q$ be a finite type scheme. Let $\overline{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ be the base-change to the algebraic closure, and recall that it is equipped with the geometric Frobenius $\mathrm{Fr}^g\colon \overline{X} \to \overline{X}$, which satisfies that the fixed-points by $\left(\mathrm{Fr}^g\right)^r$ are $X\left(\mathbb{F}_{q^r}\right)$. If $x \in X\left(\mathbb{F}_{q^r}\right)$, then there is some minimal d_x such that x is in the image of $X\left(\mathbb{F}_{q^{d_x}}\right) \to X\left(\mathbb{F}_{q^r}\right)$, i.e. the minimal number such that it is fixed by $\left(\mathrm{Fr}^g\right)^{d_x}$. A closed point $|X|$ corresponds to some $\tilde{x} = \operatorname{Spec} k(x) \subseteq X$. Note that for every chosen isomorphism $k(x) \cong \mathbb{F}_{q^{d_x}}$ we get a (different) $\mathbb{F}_{q^{d_x}}$-point of X whose image is $\tilde{x}$, so like before there are d_x lifts to $X\left(\mathbb{F}_{q^r}\right)$ if $d_x \mid r$ and 0 otherwise. Also recall that if $\mathcal{G} \in \mathrm{D}^b_c(X)$ is a chain complex of sheaves, then $\overline{\mathcal{G}} \in \mathrm{D}^b_c\left(\overline{X}\right)$ has a Frobenius map $\varphi\colon \left(\mathrm{Fr}^g\right)^* \overline{\mathcal{G}} \to \overline{\mathcal{G}}$. Therefore, for a point $x\colon \operatorname{Spec} \mathbb{F}_{q^r} \to X$ we get a map $\varphi^r_x\colon \overline{\mathcal{G}}_x \to \overline{\mathcal{G}}_x$. If $\tilde{x}$ is a closed point we let $\varphi_{\tilde{x}} = \varphi^{d_x}_x$.

This is very analogues to what we had before, where the set $\overline{X}$ with $F : \overline{X} \to \overline{X}$ is replaced by the scheme $\overline{X}$ with $\mathrm{Fr}^g\colon \overline{X} \to \overline{X}$, the relationship with points is still very similar, and any sheaf over X yields a sheaf over $\overline{X}$ with $\varphi\colon \left(\mathrm{Fr}^g\right)^* \overline{\mathcal{G}} \to \overline{\mathcal{G}}$.

Definition 14 (Sheaf to function correspondence)**.** Given $\mathcal{G} \in \mathrm{D}^b_c(X)$, we define the function $T^r\mathcal{G}\colon X\left(\mathbb{F}_{q^r}\right) \to \mathbb{Q}_\ell$ by $T^r\mathcal{G}(x) = \operatorname{Tr}\left(\varphi^r_x \mid \overline{\mathcal{G}}_x\right)$.

Definition 15. For $f\colon X \to Y$ we define

1. $f^*\colon \overline{\mathbb{Q}}_\ell^{Y\left(\mathbb{F}_{q^r}\right)} \to \overline{\mathbb{Q}}_\ell^{X\left(\mathbb{F}_{q^r}\right)}$ by $f^*T(x) = T(fx)$.
2. $f_!\colon \overline{\mathbb{Q}}_\ell^{X\left(\mathbb{F}_{q^r}\right)} \to \overline{\mathbb{Q}}_\ell^{Y\left(\mathbb{F}_{q^r}\right)}$ by $f_!T(y) = \sum_{fx=y} T(x)$.

Example 16. For $Y = \operatorname{Spec} \mathbb{F}_q$ there is a single point so that $T^r\mathcal{G} = \operatorname{Tr}\left(\varphi^r \mid \overline{\mathcal{G}}\right)$.

Furthermore note that if $x\colon \operatorname{Spec} \mathbb{F}_{q^r} \to X$ is an $\mathbb{F}_{q^r}$-point, than $T^r\mathcal{G}(x) = x^*T^r\mathcal{G} = T^r\mathcal{G}_x$.

Example 17. For $f\colon X \to \operatorname{Spec} \mathbb{F}_q$ we get $f_!T^r\mathcal{G} = \sum_{x \in X\left(\mathbb{F}_{q^r}\right)} \operatorname{Tr}\left(\varphi^r_x \mid \overline{\mathcal{G}}_x\right)$.

Proposition 18. *Let $f\colon X \to Y$ be a morphism, then $f^*T^r = T^r f^*$.*

Proof. Immediate. One quick way to see this is by employing the case for a point $T^r f^* \mathcal{G}(x) = T^r x^* f^* \mathcal{G} = T^r (px)^* \mathcal{G} = (fx)^* T^r \mathcal{G} = x^* f^* T^r \mathcal{G} = f^* T^r \mathcal{G}(x)$. □

Theorem 19 (Grothendieck-Lefschetz trace formula)**.** *Let $f\colon X \to Y$ be a separated morphism, then $f_! T^r = T^r f_!$.*

This theorem is not obvious, and definitely much harder than the case for sets. We will discuss the proof in the end of the talk.

Corollary 20. *Let $X/\mathbb{F}_q$ and $\mathcal{G} \in \mathrm{D}_c^b(X)$, then we have*

$$\sum_{x \in X(\mathbb{F}_{q^r})} \mathrm{Tr}\left(\varphi_x^r \mid \overline{\mathcal{G}}_x\right) = \mathrm{Tr}\left(\varphi^r \mid \mathrm{H}_c^*\left(\overline{X}; \overline{\mathcal{G}}\right)\right).$$

In particular, for the constant sheaf $\mathcal{G} = f^* \mathbb{Q}_\ell$ we get

$$\# X(\mathbb{F}_{q^r}) = \mathrm{Tr}\left(\varphi^r \mid \mathrm{H}_c^*\left(\overline{X}; \mathbb{Q}_\ell\right)\right).$$

Example 21. For $X = \mathbb{P}^n$, by decomposing it as $\mathbb{P}^n = \mathbb{P}^{n-1} \cup \mathbb{A}^n$ we get inductively that $|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + \cdots + q^n$. Additionally, we know that $\mathrm{H}_c^{2i}(\mathbb{P}^n; \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-i)$ for $i = 0, \ldots, n$ and 0 otherwise, so that φ acts on H_c^{2i} by multiplication by q^i, thus $\mathrm{Tr}(\varphi \mid \mathrm{H}_c^*(\mathbb{P}^n; \mathbb{Q}_\ell)) = 1 + q + \cdots + q^n$.

Definition 22. Given $\mathcal{G} \in \mathrm{D}_c^b(X)$, we define the L-function again using local L-functions

$$L(X, \mathcal{G}, t) := \prod_{\tilde{x} \in |X|} L\left(\varphi_{\tilde{x}}, t^{d_{\tilde{x}}}\right) = \prod_{\tilde{x} \in |X|} \det\left(1 - t^{d_{\tilde{x}}} \varphi_{\tilde{x}}\right)^{-1}.$$

Proposition 23. $L(X, \mathcal{G}, t) = \exp\left(\sum_{r=1}^{\infty} \left(\sum_{x \in X(\mathbb{F}_{q^r})} \mathrm{Tr}\left(\varphi_x^r \mid \overline{\mathcal{G}}_x\right)\right) \frac{t^r}{r}\right)$.

Proof. Exactly the same as we did for sets. □

Corollary 24 (Grothendieck-Lefschetz trace formula for L-functions)**.** *Let $f\colon X \to Y$ be a morphism, then $L(X, \mathcal{G}, t) = L(Y, f_! \mathcal{G}, t)$. In particular, for $Y = \operatorname{Spec} \mathbb{F}_q$ we get $L(X, \mathcal{G}, t) = \det\left(1 - t\varphi \mid \mathrm{H}_c^*\left(\overline{X}; \overline{\mathcal{G}}\right)\right)^{-1}$.*

Remark 25. Once more, the Grothendieck-Lefschetz trace formula is a local-to-global kind of a thing, we compute something defined locally (the L-function, as the product of the local L-functions) via something defined globally (the cohomology $\mathrm{H}_c^*\left(\overline{X}; \overline{\mathcal{G}}\right)$).

4 Implication of the Weil Conjectures (except RH)

Let $X/\mathbb{F}_q$ be a geometrically connected smooth and projective variety of dimension n, and we are interested in the amount of $\mathbb{F}_{q^r}$ points for all r, which we encoded in a generating function.

Definition 26. The *zeta function* of $X/\mathbb{F}_q$ is defined to be

$$Z(X,t) := \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})| \frac{t^r}{r}\right) \in \mathbb{Q}[[t]],$$

Using 23 and the fact that on the constant sheaf the Frobenius acts by 1, we get that $Z(X,t) = L(X, \mathbb{Q}_\ell, t)$. Now, applying the Grothendieck-Lefschetz trace formula, and remembering that in the proper case $\mathrm{H}_c^* = \mathrm{H}^*$, we get

Corollary 27 (Grothendieck-Lefschetz trace formula for zeta functions)**.** *Let $X/\mathbb{F}_q$ be as above, then*

$$Z(X,t) = \det\left(1 - t\varphi \mid \mathrm{H}^*\left(\overline{X}; \mathbb{Q}_\ell\right)\right)^{-1} = \prod_{i=0}^{2n} P_i(t)^{(-1)^{i+1}}$$

where $P_i(t) = \det\left(1 - t\varphi \mid \mathrm{H}^i\left(\overline{X}; \mathbb{Q}_\ell\right)\right)$.

With this, we have all the technology to prove the Weil conjectures except for the Riemann Hypothesis, so let's recall them.

Theorem 28 (Weil conjectures)**.** *Let $X/\mathbb{F}_q$ be as above, then:*

1. *Rationality: $Z(X,t)$ is a rational function.*
2. *Functional Equation: $Z\left(X, \frac{1}{q^n t}\right) = \pm q^{\frac{n\chi}{2}} t^{\chi} Z(X,t)$, where χ is the Euler characteristic of X.*
3. *Riemann Hypothesis: we have $P(t) = 1 - t$, $P_{2n}(t) = 1 - q^{2n}t$ and $P_i(t) = \prod_j (1 - \alpha_{ij}t)$ with $|\alpha_{ij}| = q^{\frac{i}{2}}$.*

Proof of Weil conjectures except RH. The Grothendieck-Lefschetz trace formula already admits $Z(X,t)$ as a rational function.

The functional equation follows from Poincare duality $\mathrm{H}^i = \left(\mathrm{H}^{2n-i}\right)^{\vee}(-n)$ (which stems from the 6 functor formalism and the fact that the dualizing sheaf $f^!\mathbb{Q}_\ell$ is an invertible object, and that in the proper case $\mathrm{H}_c^* = \mathrm{H}^*$ and $\mathrm{H}_*^{\mathrm{BM}} = \mathrm{H}_*$). In particular, we have an equality of dimensions $\beta_i = \dim \mathrm{H}^i = \dim \mathrm{H}^{2n-i} = \beta_{2n-i}$, and the eigenvalues of the Frobenius are related by $\alpha_{ij} = \alpha_{2n-i,j}^{-1} q^n$ (up to permutation).

$$
\begin{aligned}
P_i\left(\frac{1}{q^n t}\right) &= \prod_{j=1}^{\beta_i}\left(1 - \frac{\alpha_{ij}}{q^n t}\right)^{(-1)^{i+1}} \\
&= \prod_{j=1}^{\beta_i}\left(1 - \frac{1}{\alpha_{2n-i,j} t}\right)^{(-1)^{i+1}} \\
&= \prod_{j=1}^{\beta_i} - \left(\alpha_{2n-i,j} t\right)^{(-1)^i} \left(1 - \alpha_{2n-i,j} t\right)^{(-1)^{2n-i+1}} \\
&= \pm \left(\prod_{j=1}^{\beta_i} \alpha_{2n-i,j}\right) t^{(-1)^i \beta_i} P_{2n-i}\left(t\right)
\end{aligned}
$$

Now using the fact that the eigenvalues at i match with those of $2n-i$ to give q^n each, and that $\chi := \sum \left(-1\right)^i \beta_i$, we get

$$
Z\left(X, \frac{1}{q^n t}\right) = \pm \prod_i \left(\left(\prod_{j=1}^{\beta_i} \alpha_{2n-i,j}\right) t^{(-1)^i \beta_i}\right)^{(-1)^i} Z\left(X, t\right) = \pm q^{\frac{n\chi}{2}} t^{\chi} Z\left(X, t\right).
$$

□

5 Proof of the Grothendieck-Lefschetz Trace Formula

First, let's recall the theorem.

Definition 29 (Sheaf to function correspondence)**.** Given $\mathcal{G} \in \mathrm{D}_c^b\left(X\right)$, we define the function $T^r\mathcal{G} \colon X\left(\mathbb{F}_{q^r}\right) \to \overline{\mathbb{Q}}_\ell$ by $T^r\mathcal{G}\left(x\right) = \mathrm{Tr}\left(\varphi_x^r \mid \overline{\mathcal{G}}_x\right)$.

Theorem 30 (Grothendieck-Lefschetz trace formula)**.** *Let $f \colon X \to Y$ be a separated morphism of schemes of finite type over $\mathbb{F}_q$, then $f_! T^r = T^r f_!$.*

First note that the version for $r = 1$ implies the version for arbitrary r by replacing q by q^r and base-changing our schemes $- \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$.

Steps of the proof:

1. Reduce the theorem to the absolute case (i.e. $f \colon X \to \operatorname{Spec} \mathbb{F}_q$).
   Follows from proper base-change, and that $i^* T = T i^*$ for any morphism i.

2. Reduce the theorem to curves.
   This step uses standard algebraic geometry, one decomposes X into an open U

and its closed complement Z, with $U \to Y$ of lower dimension and fibers which are curves. By Noetherian induction the theorem is true for Z. By induction on the dimension the theorem is true for Y, and by assumption for each fiber of $U \to Y$, thus also for U. Then it is also true for X using the cofiber sequence $j_! j^* \mathcal{G} \to \mathcal{G} \to i_* i^* \mathcal{G}$.

3. Reduce the theorem to smooth curves and $\mathcal{G} = \mathbb{Q}_\ell$.
   I don't understand this argument deeply. The idea is to decompose the curve into parts over which the sheaf is locally constant, find an etale cover over which it is constant, and use the case of $\mathbb{Q}_\ell$.

4. Prove the case of a curve and $\mathcal{G} = \mathbb{Q}_\ell$.

5.1 Lefschetz Trace Formula

In the end, we are reduced to showing $f_! T = T f_!$ for $f \colon X \to \operatorname{Spec} \mathbb{F}_q$ where X is a smooth curve. In this case of course $\sum_{x \in X(\mathbb{F}_q)} \operatorname{Tr}(\varphi_x \mid \mathbb{Q}_\ell) = \# X(\mathbb{F}_q)$, so we need to show $\# X(\mathbb{F}_q) = \operatorname{Tr}\left(\varphi \mid \mathrm{H}^*\left(\overline{X}; \mathbb{Q}_\ell\right)\right)$.

Recall that $X(\mathbb{F}_q)$ are the fixed points of Frobenius on X. In fact, one can state a more general theorem, which is very similar to the topological Lefschetz Trace Formula, so let's first recall the topological version.

Let M be a closed n-manifold, and let $F \colon M \to M$ a self-map. Consider the manifold $M \times M$, and the two sub-manifolds $\Delta = \Gamma_{\mathrm{id}}$ and Γ_F (graph of F), which induce classes $[\Gamma_F] \in \mathrm{H}_n(\Gamma_F; \mathbb{Q}) \to \mathrm{H}_n(M \times M; \mathbb{Q})$. Consider their intersection number $[\Delta] \cdot [\Gamma_F] := [\Delta]^* \smile [\Gamma_F]^* \in \mathrm{H}^{2n}(M \times M; \mathbb{Q}) = \mathbb{Q}$.

Theorem 31. *Let M and $\phi \colon M \to M$ be as above, then*

$$[\Delta] \cdot [\Gamma_F] = \operatorname{Tr}(F \mid \mathrm{H}^*(M; \mathbb{Q})).$$

In particular, if ϕ has only finitely many fixed-points then

$$\sum_{x \in M^\phi} \operatorname{index}(F, x) = \operatorname{Tr}(F \mid \mathrm{H}^*(M; \mathbb{Q})).$$

In the algebro-geometric setting one can prove an analogue for this theorem. The hard part is defining the classes $[\Gamma_F]$ in $\mathrm{H}^n\left(\overline{X} \times \overline{X}; \mathbb{Q}_\ell\right)$ and showing that they satisfy similar formal properties to the case in topology. Once this is established, the same proof from topology works to show

Theorem 32. *Let $\overline{X}/\overline{\mathbb{F}}_q$ be a projective variety and $F \colon \overline{X} \to \overline{X}$ be a regular morphism, then*

$$[\Delta] \cdot [\Gamma_F] = \operatorname{Tr}\left(F \mid \mathrm{H}^*\left(\overline{X}; \mathbb{Q}_\ell\right)\right).$$

As corollary, if $\overline{X}/\overline{\mathbb{F}}_q$ comes from $X/\mathbb{F}_q$ and $F = \mathrm{Fr}^g$ is the geometric Frobenius, then it has finitely many fixed points, and one can show that the Frobenius is transversal to the identity, so that the index is 1 everywhere. Therefore we get that $\# X(\mathbb{F}_q) = [\Delta] \cdot [\Gamma_{\mathrm{Fr}^g}] = \mathrm{Tr}\left(\varphi \mid \mathrm{H}^*\left(\overline{X}; \mathbb{Q}_\ell\right)\right)$, which is what we wanted.

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