Chromatic Seminar
Chromatic Homotopy, Nilpotence, and the Thick Subcategory Theorem

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This seminar will focus on computaional results. This talk will be a little different. In the first part we will give a brief overview of chromatic homotopy, from a fairly abstract point, to understand the role of the various objects in the theory. Then, we will state the Nilpotence theorem, one of the major (and hardest) results in chromatic homotopy. Lastly, we will deduce the thick subcategory theorem from it.

1 A Brief Overview of Chromatic Homotopy

One of the classical ideas in algebraic topology is to approximate spaces or spectra by some more accessible algebraic object. There is a trade-off between having good approximation, and accessible or conceptual approximation. For example given a spectrum E ("homology theory") we can construct a functor $E_*(-) : \mathrm{Sp} \to \mathrm{GrAb}$. If $E = \mathbb{S}$ we get the homotopy groups which are a great approximation but very hard to compute, while if $E = H\mathbb{Q}$ we get very little information but it is very easy to compute. In fact, we note that E_*X is an E_*-module, i.e. $E_*(-) : \mathrm{Sp} \to \mathrm{Mod}_{E_*}$ which might give a better approximation to Sp. Moreover, if E is a ring spectrum (good, E_*E is flat etc.). Then, E_*X has some extra structure, namely the structure of an $\pi_*(E \otimes E) = E_*E$-comodule (over E_*), so the functor is $E_*(-) : \mathrm{Sp} \to \mathrm{Comod}_{(E_*, E_*E)}$, which be an even better approximation.

As we said, there is a tradeoff between how good the approximation is, and how accessible or conceptual it is. There is case (and variants of it), which are the topic of chromatic homotopy theory, which turn out to give both a very good approximation, and a very conceptual one.

Recall that a complex orientation on a ring E (i.e. $M\mathrm{U} \to E$) gives rise to a formal group law on E_* (i.e. $L \to E_*$). Moreover, if F is another complex oriented ring, then $E_*F = \pi_*(E \otimes F)$ carries formal group laws (coming from the two maps $M\mathrm{U} \to E \xrightarrow{\sim} E \otimes \mathbb{S} \to E \otimes F$ and $M\mathrm{U} \to F \xrightarrow{\sim} \mathbb{S} \otimes F \to E \otimes F$) and an isomorphism between them (since this comes from the induced map on cohomology from $\mathrm{BU}(1) \times \mathrm{BU}(1) \to \mathrm{BU}(1)$). Quillen's theorem says that the formal group law on $M\mathrm{U}_*$ is the *universal formal group law* (i.e. $L \xrightarrow{\sim} M\mathrm{U}_*$), and furthermore that $M\mathrm{U}_*M\mathrm{U}$ is the universal ring with two formal group laws and an isomorphism between them. This allows us to interpret $(M\mathrm{U}_*, M\mathrm{U}_*M\mathrm{U})$ as representing the stack of formal groups $\mathcal{M}_{\mathrm{fg}}$, that is $\mathrm{Comod}_{(M\mathrm{U}_*, M\mathrm{U}_*M\mathrm{U})} = \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$. Thus we can interpert our situation as $M\mathrm{U}_*(-) : \mathrm{Sp} \to \mathrm{QCoh}(\mathcal{M}_{\mathrm{fg}})$.

This is an approxmation of spectra by quasi-coherent sheaves on a well understood object (which is interesting for other reasons in number theory and algebraic geometry). This approximation turns out to be very good, and is able to give a lot of information about spectra (although not perfect, for example there are some non-zero spectra mapped to 0 under this approximation).

To simplify the situation, one can work p-locally, to obtain a functor $M\mathrm{U}_{(p)}(-) : \mathrm{Sp}_{(p)} \to \mathrm{QCoh}\left(\mathcal{M}_{\mathrm{fg},(p)}\right)$. One could also work globally, and the results later would work globally as well.

Now, recall that formal group laws have a notion of *height*, which is a number in $0, 1, \ldots, \infty$. One can look at the open sub-stack $\mathcal{M}_{\mathrm{fg}}^{\leq n} \subseteq \mathcal{M}_{\mathrm{fg},(p)}$, for some height n, so by restriction we get $\mathrm{Sp}_{(p)} \to \mathrm{QCoh}\left(\mathcal{M}_{\mathrm{fg}}^{\leq n}\right)$. This filtration on $\mathcal{M}_{\mathrm{fg}}$ gives a filtration on Sp called the *height filtration*, and allows us to study spectra height-by-height, in an inductive manner, a concept we will see soon in the talk on the chromatic convergence theorem. The functor above (well, not exactly it, but something very close) can be implemented in spectra, namely, there is a ring spectrum called *Morava E-Theory*, or the *Lubin-Tate spectrum*, at height n (for $0 \leq n < \infty$), denoted E_n, which sees information of height $\leq n$. This spectrum has coefficients $(E_n)_* = W(k)[[u_1, \ldots, u_{n-1}]]\left[u^{\pm 1}\right]$ where $\deg u = 2$. Specifically, E_n is constructed using LEFT, i.e. $(E_n)_*(X) = M\mathrm{U}_*(X) \otimes_{M\mathrm{U}_*} W(k)[[u_1, \ldots, u_{n-1}]]\left[u^{\pm 1}\right]$.

Furthermore, we can try an understand height n exactly, i.e. the restriction to the sub-stack $\mathcal{M}_{\mathrm{fg}}^n = \mathcal{M}_{\mathrm{fg}}^{\leq n} \setminus \mathcal{M}_{\mathrm{fg}}^{\leq n-1}$. It is known that $\mathcal{M}_{\mathrm{fg}}^n$ has a unique point, i.e. over $\overline{\mathbb{F}}_p$ there is a unique formal group law of height n, but it has a lot of automorphisms. The group of automorphisms, called the *Morava Stabilizer Group*, is denoted by $\mathbb{G}_n$. Thus $\mathcal{M}_{\mathrm{fg}}^n = \mathrm{Spec}\,\overline{\mathbb{F}}_p // \mathbb{G}_n$. The height n information can also be seen in spectra. There is a spectrum called *Morava K-Theory* at height n, denote by $K(n)$, which sees information at height n. It has coefficients $K(n)_* = \mathbb{F}_p\left[v_n^{\pm 1}\right]$ where $\deg v_n = 2(p^n - 1)$. By convention $K(0) = H\mathbb{Q}$ and $K(\infty) = H\mathbb{F}_p$. These spectra are in some senses the fields/atoms of spectra, we will see an instance of this idea in the thick subcategory theorem.

To conclude, we have spectra $M\mathrm{U}_{(p)}$, E_n and $K(n)$, which correspond to all heights, height $\leq n$ and height n. It should be noted that the first two are E_∞-rings, while $K(n)$ is known to not admit the structure of an E_∞-ring.

These spectra have lots of relationships between them, for example, for a spectrum X, $(E_n)_* (X) = 0$ if and only if $K(i)_* (X) = 0$ for $0 \leq i \leq n$ (that is, their Bousfield classes are the same, and are the height n information). (Warning, it is NOT true that $M\mathrm{U}$ and $\oplus_{i=0}^{\infty} K(i)$ are Bousfield equivalent).

2 The Nilpotence Theorem

One can ask how good is the chromatic approximation. For example, one may wonder if $x \in \pi_m X$ (i.e. $x : \mathbb{S}^m \to X$) can be detected using it, i.e. is it true that x is zero if and only if its image under Hurewicz in $M\mathrm{U}_m X$ is zero. This is not true, but still a very strong result is true. To state it we first make a definition.

Definition 1. Let $\{E^\alpha\}$ be a collection of ring spectra. We say that they *detect nilpotents* if one of the following conditions holds:

1. Let $F \xrightarrow{f} X$ be a map from a finite spectrum (the main example is $F = \mathbb{S}$) to a p-local spectrum. If the induced map $F \to X \to X \otimes E^\alpha$ is zero for all α, then $f^{\otimes n} : F^{\otimes n} \to X^{\otimes n}$ is zero for n large enough.
2. Let R be a p-local ring spectrum and $x \in \pi_m R$. If the image of x in $E^\alpha_m (R)$ is zero for all α, then x is nilpotent in $\pi_* R$.

Remark 2. We multiply $x \in \pi_m R$ and $y \in \pi_k R$ by $\mathbb{S}^{m+k} = \mathbb{S}^m \otimes \mathbb{S}^k \xrightarrow{x \otimes y} R \otimes R \xrightarrow{\text{multiplication}} R$ living in $\pi_{m+k} R$.

Devinatz, Hopkins and Smith proved the following theorem (hard!):

Theorem 3 (Nilpotence)**.** *The spectrum $M\mathrm{U}$ detects nilpotents. Similarly, $\{K(n)\}_{0 \leq n \leq \infty}$ detect nilpotents.*

This theorem shows one way in which chromatic homotopy gives control over spectra. The rest of the lecture will draw conclusions from this theorem.

Corollary 4 (Nishida)**.** *All $x \in \pi_m \mathbb{S}$ for $m \geq 1$ are nilpotent.*

Proof. By Serre's theorem, x is torsion (note that torsion means additively, and nilpotent means multiplicatively). Therefore the image of x in $M\mathrm{U}_m (\mathbb{S}) = M\mathrm{U}_m$ is torsion. But by Quillen's theorem $M\mathrm{U}_* = L$, and by Lazard's theorem $L \cong \mathbb{Z}[x_1, x_2, \ldots]$, which has no torsion. Therefore the image of x is 0, so by the Nilpotence theorem x itself is nilpotent. □

Corollary 5. *Let R be a (non-zero) p-local ring spctrum. Then $R \otimes K(n) \neq 0$ for some n.*

Proof. Assume by negation that $R \otimes K(n) = 0$ for all n. Then $1 \in \pi_0 R$ is mapped to 0 in $K(n)_0 R = 0$. Therefore by the Nilpotence theorem $1 \in \pi_0 R$ is nilpotent, which means that $1 = 0$, which means that R is the zero ring. □

3 The Thick Subcategory Theorem

I want to start with a definition. Fix some symmetric monoidal stable ∞-category $\mathcal{C}$.

Definition 6. A full subcategory $\mathcal{T} \subseteq \mathcal{C}$ is called *thick* if:

1. $0 \in \mathcal{T}$,
2. $\mathcal{T}$ is closed under cofibers,
3. $\mathcal{T}$ is closed under retracts.

There are various motivations for studying thick subcategories.

One motivation is as follows. Many properties in mathematics are closed under the above operations. Therefore, if we can show that some $X \in \mathcal{C}$ satisfies the property, then we know that the whole thick subcategory generated from X satisfies the property. If we know all subcategories $\mathcal{C}$, this can be a good way to prove theorems about them.

Another motivation is the Balmer spectrum. I don't have time for it, but if R is a (usual) ring, take $\mathcal{C} = \mathrm{Ch}^{\mathrm{perf}}(R)$ the category of chain complex of projective finitely generated modules. It turns out that if look at the collection of thick subcategories satisfying some extra conditions (called being prime ideal), then this recovers $\operatorname{Spec} R$. Thus, one can apply this for other interesting ∞-categories.

In our case we wish to study thick subcategories of $\mathcal{C} = \mathrm{Sp}_{(p)}^{\mathrm{fin}}$ i.e. p-local finite spectra. For every $0 \leq n \leq \infty$ we define $\mathcal{C}_{\geq n} = \left\{X \in \mathrm{Sp}_{(p)}^{\mathrm{fin}} \mid \forall m < n : K(m)_*(X) = 0\right\}$. For example, $\mathcal{C}_{\geq 0} = \mathrm{Sp}_{(p)}^{\mathrm{fin}}$, $\mathcal{C}_{\geq 1} = \{X \mid H\mathbb{Q}_*(X) = 0\} = \mathrm{Sp}_{\mathrm{tor}}^{\mathrm{fin}}$. Also, clearly $\mathcal{C}_{\geq n} \supseteq \mathcal{C}_{\geq n+1}$, and in fact this is always a proper subcategory (we will see this later in the seminar, as a corollary of the periodicity theorem).

Remark 7. The Atiyah-Hirzebruch spectral sequence says that for a spectrum X and a spectrum A, we have a spectral sequence $E_{p,q}^2 = H_p(X; A_q) \Rightarrow A_{p+q}(X)$. That is we can compute the A-homology of X, from its ordinary homology with various coefficients (and a lot of differentials, in general).

Proposition 8. *$\mathcal{C}_{\geq\infty} = 0$, that is if $K(n)_*(X)$ for all $n < \infty$, then $X = 0$.*

Proof. Assume that X is not zero. Then by universal coefficient theorem (since X is p-local) $H_*(X, \mathbb{F}_p) \neq 0$. But, since X is finite, $H_m(X, \mathbb{F}_p) = 0$ for $m \geq m_0$. We now use the Atiyah-Hirzebruch spectral sequence for $K(n)$. We have $E_{p,q}^2 = H_p\left(X; K(n)_q\right) \Rightarrow K(n)_{p+q}(X)$. Recall that $K(n)_* = \mathbb{F}_p\left[v_n^{\pm 1}\right]$ where $\deg v_n = 2(p^n - 1)$. Thus, for q multiple of $\deg v_n$, the E^2 -page is the row $H_*(X, \mathbb{F}_p)$, and otherwise 0. This line is bounded between 0 and m, so if we take n large enough, that is $2(p^n - 1) > m_0 + O(1)$, then all differentials either start or end at 0. Therefore the spectral sequence collapses, and we get $K(n)_*(X) = H_*(X, \mathbb{F}_p)\left[v_n^{\pm 1}\right] \neq 0$. $\square$

Furthermore, one can prove that for a finite spectrum X, if $K(n)_*(X) = 0$ then $K(n-1)_*(X) = 0$ as well. Therefore, in fact $\mathcal{C}_{\geq n} = \{X \mid K(n-1)_*(X) = 0\}$. A spectrum in $\mathcal{C}_{\geq n}$ is called of *type* $\geq n$, and if $K(n)_*(X) \neq 0$ it is of *type* n.

The main theorem is then:

Theorem 9 (Thick Subcategory)**.** *The thick subcategories of* $\mathrm{Sp}_{(p)}^{\mathrm{fin}}$ *are the* $\mathcal{C}_{\geq n}$ *for* $0 \leq n \leq \infty$*.*

Proof. Let $\mathcal{T}$ be some thick subcategory. First of all we note that if $X \in \mathcal{T}$, then for any $Y \in \mathrm{Sp}_{(p)}^{\mathrm{fin}}$, $X \otimes Y \in \mathcal{T}$. This is because $X \otimes \mathbb{S} = X \in \mathcal{T}$, and every Y is built by extensions from $\mathbb{S}$, so because $\mathcal{T}$ is thick, $X \otimes Y \in \mathcal{T}$.

If $\mathcal{T} = 0$ then $\mathcal{T} = C_{\geq \infty}$, and we are done, so from now on assume that $\mathcal{T} \neq 0$. Take $X \in \mathcal{T}$ of minimal type, say n, i.e. $X \in \mathcal{C}_{\geq n} \setminus \mathcal{C}_{\geq n+1}$. We have $\mathcal{T} \subseteq \mathcal{C}_{\geq n}$, for otherwise there is $Z \in \mathcal{T} \setminus \mathcal{C}_{\geq n}$, i.e. of type $< n$, which contradicts the minimality of X. To finish, we show that $\mathcal{C}_{\geq n} \subseteq \mathcal{T}$.

Let $DX = \hom(X, \mathbb{S})$ be the Spanier-Whitehead dual, and define $\mathbb{S} \to X \otimes DX$ be the mate of the identity under the adjunction. Define the fiber to form $F \xrightarrow{\alpha} \mathbb{S} \to X \otimes DX$. Since X is of type n, we have $K(m)_*(X) \neq 0$ for $m \geq n$, thus $K(m)_* \to K(m)_*(X) \otimes_{K(m)_*} K(m)_*(X)^*$ is an injection (they are graded finite dimensional vector spaces over $\mathbb{F}_p$). Therefore, the kernel $K(m)_*(F) \xrightarrow{\alpha_*} K(m)_*$ is zero.

Now let $Y \in \mathcal{C}_{\geq n}$, and we show that $Y \in \mathcal{T}$. Similarly to before, look at $\mathbb{S} \xrightarrow{\beta} Y \otimes DY$. Pre-compose this with $F \xrightarrow{\alpha} \mathbb{S}$, to get $f = \beta\alpha$ (note that F was formed from X). For $m \geq n$, we had $\alpha_* = 0$, thus $f_* = \beta_*\alpha_* = 0$. For $m < n$, we have $K(m)_*(Y) = 0$, thus the target of β_* is 0, so $f_* = \beta_*\alpha_* = 0$.

We have shown that f_* for every m, so from the Nilpotence theorem we get that $f : F \to Y \otimes DY$ is nilpotent. That is, there is k such that $f^{\otimes k} = 0$, i.e. $F^{\otimes k} \xrightarrow{\alpha^{\otimes k}} \mathbb{S}^{\otimes k} \xrightarrow{\beta^{\otimes k}} (Y \otimes DY)^{\otimes k}$ is zero. The spectrum $Y \otimes DY$ also has an evaluation map $Y \otimes DY \to \mathbb{S}$. Compose with it $k-1$ times, we get that $F^{\otimes k} \to (Y \otimes DY)^{\otimes k} \to Y \otimes DY$ is zero, and taking the mate we get that $F^{\otimes k} \otimes Y \xrightarrow{\alpha^{\otimes k} \otimes \mathrm{id}} \mathbb{S} \otimes Y = Y$ is zero. Therefore, the cofiber of this map is $C\left(\alpha^{\otimes k} \otimes \mathrm{id}\right) = Y \oplus \Sigma\left(F^{\otimes k} \otimes Y\right)$. Recall that we wanted to show that $Y \in \mathcal{T}$, but $\mathcal{T}$ is closed under retracts, so it is enough to show that $C\left(\alpha^{\otimes k} \otimes \mathrm{id}\right) \in \mathcal{T}$.

The map $\alpha^{\otimes k} \otimes \mathrm{id}$ is the composition $F^{\otimes k} \otimes Y \xrightarrow{\alpha \otimes \mathrm{id}} F^{\otimes k-1} \otimes Y \xrightarrow{\alpha \otimes \mathrm{id}} \cdots \xrightarrow{\alpha \otimes \mathrm{id}} Y$, so to show that $C\left(\alpha^{\otimes k} \otimes \mathrm{id}\right) \in \mathcal{T}$, it is enough to show that

$$\mathcal{T} \ni C\left(F^{\otimes l} \otimes Y \xrightarrow{\alpha \otimes \mathrm{id}} F^{\otimes l-1} \otimes Y\right) = C(F \to \mathbb{S}) \otimes F^{\otimes l-1} \otimes Y = X \otimes DX \otimes F^{\otimes l-1} \otimes Y$$

We know that $X \in \mathcal{T}$, thus the tensor with it is in $\mathcal{T}$, so we are done. □