Chromatic Seminar
Topological Modular Forms and $\pi_*\mathrm{TMF}$

Shay Ben Moshe

11/09/2019

There are two new references which are really great, they are more of a survey, but they give a very nice treatment of all of the material here [Beh19; Mei17]. There are notes by Akhil [Mat12]. The original computation, which is fairly well written, is in [Bau03]. Another computation from roughly the same time is in [Rez01]. Supplementary material on the moduli stack of elliptic curves is in [MO17]. Another good survey of the theory, which doesn't really get to the computations, is [Goe09].

1 What's Special About Height 1 and a Construction of KO

Recall that the one of the main objects of study in chromatic homotopy theory is E_n, Morava E-Theory at height n and prime p. However, height $n = 1$ is very special for various reasons, e.g.:

1. We have an integral model, namely complex K-theory KU, such that $E_1 = \mathrm{KU}_p^\wedge$ is its p-completion.
2. We have a geometric model, i.e. $E_1^*(X)$ has a geometric definition in terms of X, namely vector bundles on X (which for example allows the construction of *equivariant* Morava E-theory, using KU_G i.e. G-vector bundles).

There are some programs trying to address these in higher heights, but in this talk, whose topic is TMF, we will only address the first point, namely the existence of an integral model at height 2. In fact, the integral model will be analogues to KO i.e. orthogonal K-theory (of vector bundles over the real numbers), which is KU^{hC_2}, rather than the whole of KU.

To generalize the construction of KO to height 2, we give an alternative construction of it, which will be more amenable to generalizations. A reference for this part is in [Mat12, Section 2].

Definition 1. Let X be a scheme. A *one dimensional torus* over X is a group scheme $\mathbb{G}/X$ that is isomorphic to $\mathbb{G}_m$ after a faithfully flat extension. We let $\mathcal{M}_{1\text{tori}} : \text{Sch}^{\text{op}} \to \text{Grpd}$ be the functor sending X to the groupoid of tori over X with isomorphisms between them. This is the *moduli stack of (one dimensional) tori.*

Theorem 2. *There is an isomorphism of stacks* $\mathcal{M}_{1\text{tori}} \cong \text{B}C_2$*, meaning that* $\mathcal{M}_{1\text{tori}}(X) = \hom(X, \text{B}C_2)$.

We will not prove this, but we notice that the automorphisms of $\mathbb{G}_m = \text{Spec}\,\mathbb{Z}\left[t, t^{-1}\right]$ (as a group scheme) are the identity and $x \mapsto x^{-1}$. The stack $\mathcal{M}_{1\text{tori}}$ has a structure sheaf $\mathcal{O}$ (in the etale topology). Furthermore, there is a morphism $\mathcal{M}_{1\text{tori}} \to \mathcal{M}_{\text{fg}}$ sending a group scheme $\mathbb{G}$ to the formal group which is the completion at the identity $\mathbb{G}_e^{\wedge}$. Notice that such formal groups have height ≤ 1. From any formal group $\widehat{\mathbb{G}}$ we can form $\omega_{\mathbb{G}^{\wedge}} = T_e^*\widehat{\mathbb{G}}$ the cotangent bundle which also parameterizes invariant bundles. This assemble to the *sheaf of invariant differential forms* ω on $\mathcal{M}_{\text{fg}}$, which can be pulled to $\mathcal{M}_{1\text{tori}}$.

Theorem 3. *There exists a sheaf of* $\mathbb{E}_\infty$*-rings* $\mathcal{O}^{\text{top}}$ *on* $\mathcal{M}_{1\text{tori}}$*. For an etale map* $\mathbb{G} : \text{Spec}\,R \to \mathcal{M}_{1\text{tori}}$ *denote the* $\mathbb{E}_\infty$*-ring* $E_{\mathbb{G}} = \mathcal{O}^{\text{top}}(\text{Spec}\,R)$*. Then the following holds:*

1. $E_{\mathbb{G}}$ *is even-periodic (in particular complex-orientable).*
2. $\pi_0 E_{\mathbb{G}} = \mathcal{O}(\text{Spec}\,R) = R$*, and* $\pi_t E_{\mathbb{G}} = \omega_{\mathbb{G}}^{\otimes t/2}$ *(0 for odd).*
3. $\text{Spf}\, E_{\mathbb{G}}^0(\text{BU}(1)) = \mathbb{G}_e^{\wedge}$.

We can then take the global sections $\Gamma(\mathcal{M}_{1\text{tori}}; \mathcal{O}^{\text{top}})$ (as an $\mathbb{E}_\infty$-ring). We have a descent spectral sequence (coming from taking homotopy groups) $H^s(\mathcal{M}_{1\text{tori}}; \pi_t\mathcal{O}^{\text{top}}) \Rightarrow \pi_{t-s}\Gamma(\mathcal{M}_{1\text{tori}}; \mathcal{O}^{\text{top}})$. By the above we know that $H^s(\mathcal{M}_{1\text{tori}}; \pi_t\mathcal{O}^{\text{top}}) = H^s\left(\text{B}C_2; \omega^{\otimes t/2}\right)$. One can then show that this spectral sequence coincides with the KU-based Adams spectral for computing the homotopy groups of KO, and we get that $\text{KO} = \Gamma(\mathcal{M}_{1\text{tori}}; \mathcal{O}^{\text{top}})$.

2 The Construction of TMF

One may wonder if we can take a similar approach at height 2. For this we would need some stack of geometric objects, from which we can form formal groups with height ≤ 2. The *stack of elliptic curves* $\mathcal{M}_{\text{ell}}$ is exactly such a stack! Given an elliptic curve C over a ring R, we can consider the completion at the identity $C_e^{\wedge}$, giving a map $\mathcal{M}_{\text{ell}} \to \mathcal{M}_{\text{fg}}$. It is also known that the formal group associated to an elliptic curve is of height ≤ 2, the case of height 1 is called *ordinary*, and the case of height 2 it is called *supersingular*.

Naively, one might hope that we can use only the (sub)stack of supersingular elliptic curves for the construction, however it is highly non-connected. The idea is that we use ordinary elliptic curves to interpolate between the supersingular ones in an interesting way. In fact, $\mathcal{M}_{\mathrm{ell}}$ is not compact, and for some purposes it is better to work with the compactification $\overline{\mathcal{M}}_{\mathrm{ell}}$. This stack doesn't admit a simple description as $\mathcal{M}_{1\mathrm{tori}} \cong \mathrm{B}C_2$, but it is still not too complicated, especially away from $2, 3$.

Just like in the case of $\mathcal{M}_{1\mathrm{tori}}$, we can pullback along $\overline{\mathcal{M}}_{\mathrm{ell}} \to \mathcal{M}_{\mathrm{fg}}$ to get the sheaf of invariant differentials ω. Classically, there is the notion of a *modular form* described as a function on the upper half plane satisfying some properties, or at least this the definition over $\mathbb{C}$. From a modern perspective, a modular form (over $\mathbb{Z}$) of weight k is exactly a section of $\omega^{\otimes k}$, i.e. $\mathrm{MF}_k = H^0\left(\overline{\mathcal{M}}_{\mathrm{ell}}; \omega^{\otimes k}\right)$. This ring turns out to be fairly small, specifically $\mathrm{MF}_* = \mathbb{Z}\left[c_4, c_6, \Delta\right] / \left(12^3\Delta = c_4^3 - c_6^2\right)$ where $c_k \in \mathrm{MF}_k$ and $\Delta \in \mathrm{MF}_{12}$ (see [Rez01, Proposition 10.3]). As we can see, away from $2, 3$ this is even simpler, i.e. $\mathrm{MF}_*\left[\frac{1}{2\cdot 3}\right] = \mathbb{Z}\left[c_4, c_6\right]$.

We now have the analogues big theorem:

Theorem 4 (Goerss-Hopkins-Miller-Lurie)**.** *There exists a sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O}^{\mathrm{top}}$ on $\overline{\mathcal{M}}_{\mathrm{ell}}$. For an etale map $C : \mathrm{Spec}\, R \to \overline{\mathcal{M}}_{\mathrm{ell}}$ denote the $\mathbb{E}_\infty$-ring $E_C = \mathcal{O}^{\mathrm{top}}(\mathrm{Spec}\, R)$. Then the following holds:*

1. *E_C is even-periodic (in particular complex-orientable).*
2. *$\pi_0 E_C = \mathcal{O}(\mathrm{Spec}\, R) = R$, and $\pi_t E_C = \omega_{\mathbb{G}}^{\otimes t/2}$ (0 for odd).*
3. *$\mathrm{Spf}\, E_C^0(\mathrm{BU}(1)) = C_e^\wedge$.*

Remark 5. It might seem like this is not a very sophisticated thing. Given $C : \mathrm{Spec}\, R \to \overline{\mathcal{M}}_{\mathrm{ell}}$, we can construct a similar cohomology theory E_C using the Landweber exact functor theorem. However, for this we need some conditions on the curve (the map should be flat), and more over the theorem says that we can lift E_C from a cohomology theory to a spectrum and even to an $\mathbb{E}_\infty$-ring. Furthermore, it assembles into a sheaf (on the etale site).

Now, analogously to the case of $\mathrm{KO} = \Gamma\left(\mathcal{M}_{1\mathrm{tori}}; \mathcal{O}^{\mathrm{top}}\right)$ we define

Definition 6. $\mathrm{TMF} = \Gamma\left(\mathcal{M}_{\mathrm{ell}}; \mathcal{O}^{\mathrm{top}}\right)$, and the more complicated variant $\mathrm{Tmf} = \Gamma\left(\overline{\mathcal{M}}_{\mathrm{ell}}; \mathcal{O}^{\mathrm{top}}\right)$. Their connective cover (which turns out to be the same) is denoted by $\mathrm{tmf} = \mathrm{TMF}_{\geq 0}$. These are all $\mathbb{E}_\infty$-rings.

We will work with TMF from now on. Similarly to before, taking homotopy groups give a spectral sequence $E_2^{s,t} = H^s\left(\mathcal{M}_{\mathrm{ell}}; \pi_t \mathcal{O}^{\mathrm{top}}\right) \Rightarrow \pi_{t-s}\mathrm{TMF}$ (and similarly for $\overline{\mathcal{M}}_{\mathrm{ell}}$ and Tmf).

To connect this with modular forms, recall that $\mathrm{MF}_* = H^0\left(\overline{\mathcal{M}}_{\mathrm{ell}}; \omega^{\otimes *}\right)$, thus the edge homomorphism in the spectral sequence is $\pi_*\mathrm{Tmf} \to \mathrm{MF}_{*/2}$. As there are modular forms only in non-negative weights, it makes sense to take the map $\pi_*\mathrm{tmf} \to \mathrm{MF}_{*/2}$, and for example this map is an isomorphism away from $2,3$ (as we shall see soon), which motivates the name. We recall that since tmf is an $\mathbb{E}_\infty$-ring, it has a unit $\mathbb{S} \to \mathrm{tmf}$, which gives us $\pi_*\mathbb{S} \to \pi_*\mathrm{tmf} \to \mathrm{MF}_{*/2}$, so in some sense $\pi_*\mathrm{tmf}$ is a mixture of the two.

3 Homotopy Groups Away From $2,3$

In this part we localize everything away from $2,3$, i.e. work over $\mathbb{Z}\left[\frac{1}{2\cdot 3}\right]$, but we will sometimes omit this from the notation. We first compute $H^s\left(\mathcal{M}_{\mathrm{ell}};\pi_t\mathcal{O}^{\mathrm{top}}\right) = H^s\left(\mathcal{M}_{\mathrm{ell}};\omega^{\otimes k}\right)$ (where $k=t/2$), which is the E_2-page of our spectral sequence.

Away from the primes $2,3$ any elliptic curve is (non-canonically) isomorphic to a curve C_{c_4,c_6} in $\mathbb{P}^2$ cut by the Weierstrass equation $y^2 = x^3 - 27c_4x - 54c_6$, where the discriminant $\Delta = \frac{c_4^3 - c_6^2}{12^3}$ is invertible. Moreover, there is a canonical invariant differential on this curve given by $\eta_C = \frac{dx}{2y}$. The isomorphisms of such curves are given as follows, for every $\lambda \in \mathbb{G}_m$ there is an isomorphism $f_\lambda : C_{c_4,c_6} \xrightarrow{\sim} C_{\lambda^4 c_4,\lambda^6 c_6}$ defined by $f_\lambda(x,y) = \left(\lambda^2 x, \lambda^3 y\right)$. We also see that $f_\lambda^*\eta_C = \frac{\lambda^2 dx}{2\lambda^3 y} = \frac{1}{\lambda}\eta_C$ (Mark Behrens gets λ, am I wrong?).

To be more formal, what we see is that we have a universal elliptic curve C_{c_4,c_6} together with a trivialization of the invariant differentials over $\operatorname{Spec} A$, where $A = \mathbb{Z}\left[\frac{1}{2\cdot 3}\right]\left[c_4, c_6, \Delta^{-1}\right]$. This data is acted by $\mathbb{G}_m$, to give $\mathcal{M}_{\mathrm{ell}} = \operatorname{Spec} A//\mathbb{G}_m$, and we denote the projection by $\pi : \operatorname{Spec} A \to \mathcal{M}_{\mathrm{ell}}$. Then, to compute things like sections and cohomology, we may instead consider $\mathbb{G}_m$-equivariant versions of these on $\operatorname{Spec} A$.

Proposition 7. *Let $X = \operatorname{Spec} R$ be an affine scheme. Giving a $\mathbb{G}_m$-action on X is equivalent to giving a $\mathbb{Z}$-grading on R. In fact, we have an equivalence $\mathrm{Aff}_{\mathbb{G}_m} \cong \mathrm{GrCRing}^{\mathrm{op}}$.*

Proof. We hint the construction. A $\mathbb{G}_m$-action is a map $X \times \mathbb{G}_m \to X$, i.e. $R \to R \otimes \mathbb{Z}\left[t,t^{-1}\right] = R\left[t,t^{-1}\right]$. So given a $\mathbb{G}_m$-action we define the k-th graded piece by $R_k = \left\{r \mid r \mapsto rt^k\right\}$ (we won't show that indeed $R = \bigoplus R_k$). This works in the other, given a grading $R = \bigoplus R_k$, define an action by $r \mapsto rt^k$ for $r \in R_k$. ☐

Example 8. Let $R = k[x]$ with grading $|x| = 1$, i.e. the action comes from $x^n \mapsto x^n t^n$. We can define the sheaf $\mathcal{O}_X[1]$ to be the trivial $\mathcal{O}_X$-module, but with the action shifted by one. On the algebraic side this is the module $R[1] = k[x]\{z\}$, so that $|x^n z| = n+1$ (i.e. acted by $x^n z \mapsto x^n z t^{n+1}$) or more generally $\mathcal{O}_X[k] = (\mathcal{O}_X[1])^{\otimes k}$ given by $k[x]\left\{z^k\right\}$. The $\mathbb{G}_m$-equivariant sections of $\mathcal{O}_X[k]$, which correspond to sections on $X//\mathbb{G}_m$, are all the sections of grading 0, so that $x^n z^k$ is a section if and only if $n = -k$, i.e. $\Gamma_{\mathbb{G}_m}(X;\mathcal{O}_R[-k]) = k\left\{x^k\right\} = \Gamma^k(X;\mathcal{O}_R)$, the k-th graded piece of the non-graded global sections. Thus we have an interpretation of $\Gamma(X//\mathbb{G}_m;\mathcal{F})$.

Proposition 9. *There is an isomorphism* $H^s\left(\mathcal{M}_{\mathrm{ell}};\omega^{\otimes k}\right)=\mathbb{Z}\left[\frac{1}{2\cdot3}\right]\left[c_4,c_6,\Delta^{-1}\right]_k$ *(k-th graded piece) if* $s=0$ *and* 0 *otherwise.*

Proof. In our case, the $\mathbb{G}_m$-action on A gives it the grading $|c_k|=k$. As we have seen, $f_\lambda^*\eta_C=\frac{1}{\lambda}\eta_C$, thus we see that $\pi^*\omega$ on $\operatorname{Spec}A$ is $\mathcal{O}_A[-1]$, so we get that

$$\begin{aligned}H^s\left(\mathcal{M}_{\mathrm{ell}};\omega^{\otimes k}\right)&=H^s\left(\operatorname{Spec}A//\mathbb{G}_m;\omega^{\otimes k}\right)\\&=H^s_{\mathbb{G}_m}\left(\operatorname{Spec}A;\mathcal{O}_A[-1]^{\otimes k}\right)\\&=H^s_{\mathbb{G}_m}\left(\operatorname{Spec}A;\mathcal{O}_A[-k]\right)\\&=H^{s,k}\left(\operatorname{Spec}A;\mathcal{O}_A\right)\\&=\begin{cases}\mathbb{Z}\left[\frac{1}{2\cdot3}\right]\left[c_4,c_6,\Delta^{-1}\right]_k & s=0\\0\end{cases}\end{aligned}$$

□

Theorem 10. *([Beh19, Proposition 1.3.7], [Rez01, Proposition 15.13])* $\pi_*\mathrm{TMF}\left[\frac{1}{2\cdot3}\right]=\mathbb{Z}\left[\frac{1}{2\cdot3}\right]\left[c_4,c_6,\Delta^{-1}\right]$ *where* $\deg c_k=2k$ *(and* $\deg\Delta=12$*).*

Proof. Recall the spectral sequence $H^s\left(\mathrm{TMF};\pi_t\mathcal{O}^{\mathrm{top}}\right)\Rightarrow\pi_{t-s}\mathrm{TMF}$. We have just identified the E_2-page above, and it is all concentrated at $s=0$, so there are no differentials or extensions. Note that $\pi_t\mathcal{O}^{\mathrm{top}}=\omega^{t/2}$, hence the doubling in degrees. □

4 Homotopy Groups at 3

For references for this part, see the references in the beginning of this document.

4.1 Computing The Hopf Algebroid

As we have seen, away from $2,3$ the stack $\mathcal{M}_{\mathrm{ell}}$ was simply $\operatorname{Spec}A//\mathbb{G}_m$, which is a fairly simple description, and could be handled by $\mathbb{G}_m$-equivariant affine schemes, or $\mathbb{Z}$-graded rings. However, globally this stack is more complicated, but still admits a very concrete description. We recall that (the data of) a scheme is a functor $X:\mathrm{Aff}\to\mathrm{Set}$, and (the data of) a stack is a functor $\mathcal{M}:\mathrm{Aff}\to\mathrm{Grpd}$. Therefore, one source of stacks is as groupoid objects in schemes, or even more simply in affine schemes.

Definition 11. A *Hopf algebroid* is a groupoid object in affine schemes, though of in the algebraic side i.e. a cogroupid object in commutative rings.

To unravel what this means, we recall that a groupoid object in $\mathcal{C}$ is a pair objects $X, M \in \mathcal{C}$, together with source and target maps $s, t : M \to X$, unit map $e : X \to M$, composition $m : M \times_{X,t,s} M$ and inverse $i : M \to M$, satisfying a bunch of axioms. Then, a Hopf algebroid is the opposite thing in CRing, namely $A, \Gamma \in$ CRing together with homomorphisms $\eta_L, \eta_R : A \to \Gamma$ and so on. The corresponding stack should be though of as $\operatorname{Spec} A // \operatorname{Spec} \Gamma$.

Indeed, $\mathcal{M}_{\text{ell}}$ can be described as a Hopf algebroid. Over the ring $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ there is again a cubic $C_{a_i} : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$, i.e. every elliptic curve is pulled back from this curve. However, this pullback is not unique, thus we need to add the isomorphisms via Γ. We could write now the whole thing, but from now on we will restrict to the prime 3, which will offer some simplification.

So from now on we base change to $\mathbb{Z}_{(3)}$. Over $\mathbb{Z}_{(3)}$ the cubic from above admits a (canonical!) change of variables which eliminates all a_i's except a_2, a_4. I.e. we can work with $A = \mathbb{Z}_{(3)}[a_2, a_4]$ and the curve $C_{a_4,a_6} : y^2 = x^3 + a_2 x^2 + a_4 x$. Now we need to describe the isomorphisms of such curves. The isomorphisms fall into two types. The first is like we had before, for every $\lambda \in \mathbb{G}_m$ we have $f_\lambda : C_{a_2,a_4} \xrightarrow{\sim} C_{\lambda^2 a_2, \lambda^4 a_4}$ given by $f_\lambda(x, y) = (\lambda^2 x, \lambda^3 y)$. To handle these, instead of encoding them in Γ, we can again work in the graded setting, i.e. with a *graded Hopf algebroid*, so $|a_i| = i$. Now, we have another isomorphism sending $x \mapsto x + r, y \mapsto y$, and what we get is:

$$\begin{aligned} y^2 &= (x+r)^3 + a_2 (x+r)^2 + a_4 (x+r) \\ &= x^3 + \underbrace{(a_2 + 3r)}_{a_2'} x^2 + \underbrace{(a_4 + 2a_2 r + 3r^2)}_{a_4'} x + (r^3 + a_2 r^2 + a_4 r) \end{aligned}$$

So to get a curve of the same form we must have the constant term 0, i.e. $r^3 + a_2 r^2 + a_4 r$, and we see that $a_2 \mapsto a_2'$ and $a_4 \mapsto a_4'$. It is not too hard to see that these are all possible isomorphisms of such curves. Therefore, we define $\Gamma = A[r] / (r^3 + a_2 r^2 + a_4 r)$ with $|r| = 2$ (to match the degrees of the other things). We then need to define the various maps, η_L is the evident map coming from the inclusion $A \to A[r] / (\cdots) = \Gamma$, and η_R sends $a_2 \mapsto a_2' = a_2 + 3r$ and similarly for a_4. Since the composition of $x \mapsto x + r$ and $x \mapsto x + r'$ is $x \mapsto x + r + r'$, one can check that the comultiplication is given by $\Psi(r) = r \otimes 1 + 1 \otimes r$, etc.

4.2 Computing The Cohomologies (a.k.a. E_2-page)

Now we are interested in computing $E_2^{s,t} = H^s(\mathcal{M}_{\text{ell}}; \pi_t \mathcal{O}^{\text{top}}) = H^s(\mathcal{M}_{\text{ell}}; \omega^{\otimes k}) = H^{s,k}(A, \Gamma)$.

We will not carry the whole computation of the cohomologies $H^{s,k}(A, \Gamma)$, but we will sketch the argument. Such cohomologies can be computed explicitly by computing resolutions and so on. However, we can also solve the problem in steps.

An *invariant ideal* is an ideal $I \leq A$, s.t. $\eta_L I = \eta_R I$. If I is an invariant ideal, then $\left(\tilde{A}, \tilde{\Gamma}\right) = (A/I, \Gamma/I)$ is also a Hopf algebroid corresponding to the substack where I vanishes. Then, there is an algebraic Bockstein spectral sequence $E_2^{s,t} = H^s\left(\tilde{A}, \tilde{\Gamma}; I^t/I^{t+1}\right) \Rightarrow H^{s+t}(A, \Gamma)$. In our case, one can easily check that $I_1 = (3), I_2 = (3, a_2), I_3 = (3, a_2, a_4)$ are invariant, so we can use these to compute the cohomology in steps.

First one explicitly computes the cohomologies modulo I_3, i.e. for the (very small) Hopf algebroid $\left(\mathbb{Z}/3, \mathbb{Z}/3[r]/r^3\right)$, using the cobar construction or minimal resolution. Then we run the spectral sequence to get the cohomologies modulo I_2, and again modulo I_1, and again for the cohomology of (A, Γ).

After doing all this we get that $H^{s,k}(A, \Gamma) = \mathbb{Z}_{(3)}\left[\alpha, \beta, \Delta^{\pm 1}, c_4, c_6\right]/I$ where $|\alpha| = (1,2), |\beta| = (2,6), |\Delta| = (0,12), |c_i| = (0,i)$ and the relations are $I = \left(3\alpha, 3\beta, \alpha^2, \alpha c_i, \beta c_i, 12^3\Delta = \left(c_4^3 - c_6^2\right)\right)$.

4.3 Running The Spectral Sequence (a.k.a. differentials)

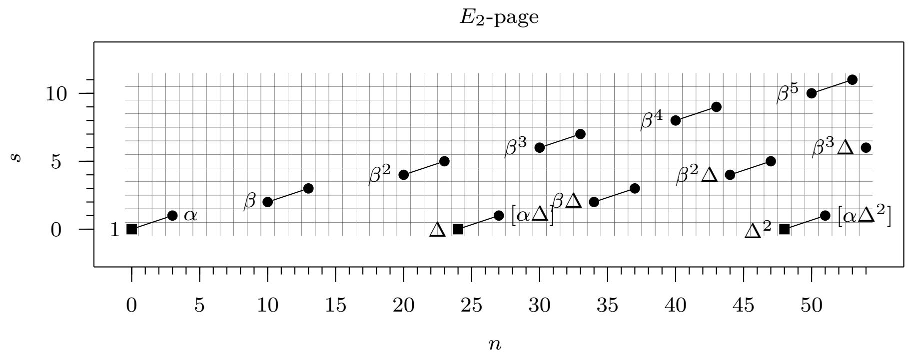
Remark 12. Our conventions for drawing the spectral sequences are as follows. We use the Adams grading. A bullet denotes $\mathbb{Z}/3$ and square denotes a $\mathbb{Z}_{(3)}$. We will see later that the spectral sequence is Δ^3-periodic, so we show only one copy and don't write the multiples by Δ^{3m}, to make the drawings clearer. Moreover, whenever we have a line going 3 to the right and 1 up, this signifies the same element multiplied by α.

Note that in particular we have a Δ-periodicity in the E_2-page. We will use the Adams grading in the spectral sequence, which gives $E_2^{s,n+s} = H^s\left(\mathcal{M}_{\text{ell}}; \pi_{n+s}\mathcal{O}^{\text{top}}\right) = H^{s,(n+s)/2}(A, \Gamma) \Rightarrow \pi_n \mathrm{TMF}_{(3)}$, and the degrees now are $\deg c_i = (0, 2i), \deg \alpha = (1,3), \deg \beta = (2,10), \deg \Delta = (0,24)$ (note that we both double the second coordinate because $\pi_t \mathcal{O}^{\text{top}} = \omega^{\otimes t/2}$, and subtract the first from second to change to Adams grading).

Lemma 13. *c_4, c_6 are permanent cycles.*

Proof idea. The idea here (which is extremely useful in other places in this theory) is to consider a moduli of more structured elliptic curves. Specifically, we consider the stack of elliptic curves with a choice of a point of order 2 (over $\mathbb{Z}_{(3)}$) $\mathcal{M}_1(2)$. Its advantage is that is much simpler, $\mathcal{M}_1(2) = \text{affine}//\mathbb{G}_m$. There is a forgetful map $f : \mathcal{M}_1(2) \to \mathcal{M}_{\text{ell}}$. It can be shown to be etale and surjective, in fact a cover. Because of this, there is a transfer map between the homotopy groups, and indeed between the spectral sequences. Because of the simple structure of $\mathcal{M}_1(2)$, it is easy to see that the spectral sequence corresponding to it is concentrated in $s = 0$, so everything is a permanent cycle. The images of c_4, c_6 from $\mathcal{M}_{\text{ell}}$ land in these elements thus they must be permanent cycles there as well. □

This allows us to work modulo c_4, c_6, which gives us $\mathbb{Z}_{(3)}\left[\alpha, \beta, \Delta^{\pm 1}\right] / \left(3\alpha, 3\beta, \alpha^2\right)$.

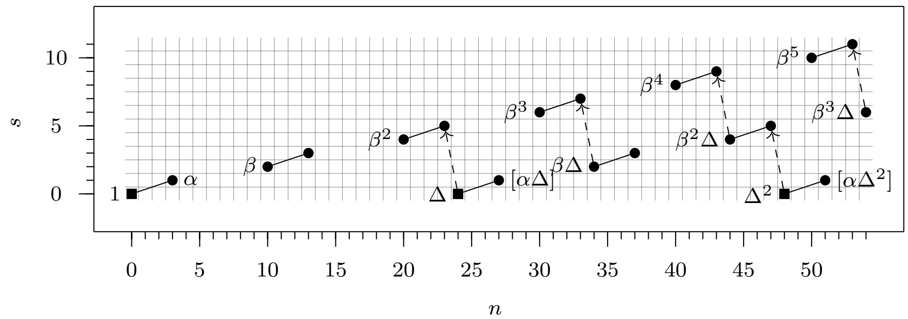


Lemma 14. *$\alpha\beta^3$ doesn't survive to the E_∞-page. Moreover, $\langle\alpha, \alpha, \alpha\rangle = \{\beta\}$.*

Proof. Recall that we have a map of stacks $\mathcal{M}_{\text{ell}} \to \mathcal{M}_{\text{fg}}$. This induces a map of spectral sequences, starting with $H^*\left(\mathcal{M}_{\text{fg}}, \omega^{\otimes *}\right) \to H^*\left(\mathcal{M}_{\text{ell}}, \omega^{\otimes *}\right)$, and converging to $\pi_*\mathbb{S} \to \pi_*\text{TMF}$. The one for the formal groups is the MU-based ANSS. There is an element $\nu \in \pi_3\mathbb{S}$, which is represented in the ANSS by an element in the E_2-page mapping to α. Similarly, there is an element β_1 mapping to β. In the homotopy groups of spheres, there is a relation called the Toda relation which is $\nu\beta_1^3 = 0$, which implies the same relation for our computation, meaning that this class doesn't survive to the E_∞-page. Similarly, the Toda bracket corresponding to the $\langle\alpha, \alpha, \alpha\rangle = \{\beta\}$ in the ANSS holds, which implies it in our case. □

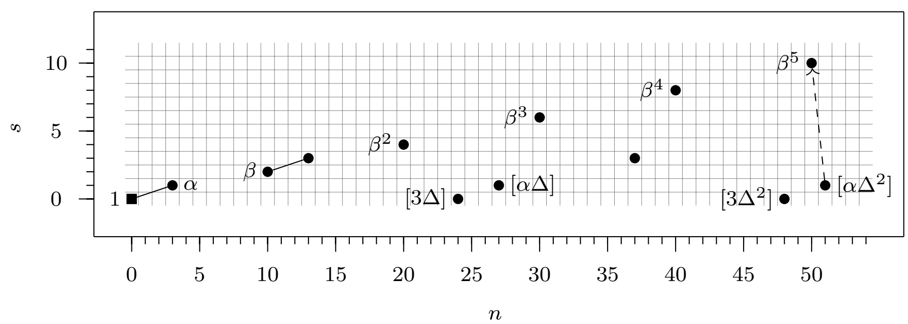
It is easy to check that the first differential that can hit anything is d_5, furthermore, this is the only differential that can hit $\alpha\beta^3$ which must vanish. Thus we get $d_5(\beta\Delta) = \pm\alpha\beta^3$. For degree reasons, $d_5(\alpha), d_5(\beta) = 0$. Then, applying the Leibniz rule we get that $d_5(\Delta) = \pm\alpha\beta^2$. Applying it inductively implies that $d_5(\Delta^n) = \pm n\alpha\beta^2\Delta^{n-1}$. From this we also immediately get $d_5\left(\beta^k\Delta^n\right) = \pm n\alpha\beta^{k+2}\Delta^{n-1}$, and since $\alpha^2 = 0$ we get $d_5\left(\alpha\beta^k\Delta^n\right) = 0$. We recall that corresponding to Δ^n is a copy of $\mathbb{Z}_{(3)}$, and at every other point we have $\mathbb{Z}/3$. This means that we still have Δ^3-periodicity (because $3 = 0$ in all the torsion groups, so these don't die). Thus, we can describe the survivors of this differential as the ring generated by $\alpha, \beta, [3\Delta], [\alpha\Delta], \left[3\Delta^2\right], \left[\alpha\Delta^2\right], \Delta^{\pm 3}$.

E_5-page

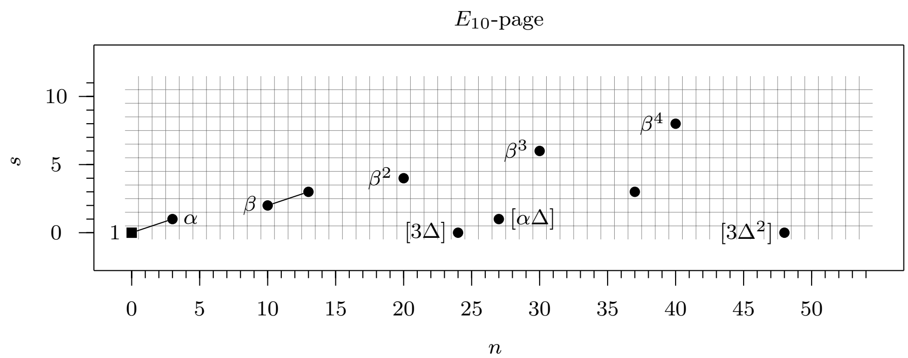


For degree reasons, we have no other differentials until d_9. We also see that α can hit anything, thus it is a permanent cycle, and so are $\beta, \beta^2, \beta^3, \beta^4$, and we denote the corresponding elements in the homotopy groups by the same letters. Now, the Toda brackets relations imply Massey product relations $\left\{\beta^3\right\} = \left\{\beta^2\beta\right\} = \beta^2 \langle \alpha, \alpha, \alpha \rangle = \left\langle \alpha\beta^2, \alpha, \alpha \right\rangle = \langle 0, \alpha, \alpha \rangle = \alpha\pi_{27}\mathrm{TMF}$. Since β^3 is non-zero, there must some element $0 \neq x \in \pi_{27}\mathrm{TMF}$ in lower filtration such that $\beta^3 = \alpha x$. The only possibility then is $x = \pm\left[\alpha\Delta^2\right]$, thus we get $\beta^3 = \pm\alpha\left[\alpha\Delta^2\right]$. Therefore, we get that $\beta^5 = \pm\alpha\beta^2\left[\alpha\Delta^2\right] = 0$ in the homotopy groups, so that β^5 doesn't survive to the E_∞-page. The only possibility is that $d_9\left(\left[\alpha\Delta^2\right]\right) = \pm\beta^5$.

E_9-page



Then, one sees that there can be no more differentials i.e. $E_{10} = E_\infty$.



Lastly, there can be no extension problems, and we conclude:

Theorem 15. $\pi_*\mathrm{TMF}_{(3)} = \mathbb{Z}_{(3)}\left[\alpha, \beta, [\alpha\Delta], [3\Delta], \left[3\Delta^2\right], \Delta^{\pm 3}, c_4, c_6\right]/J$ *where* J *is generated by the following relations:*

- $\frac{12^3}{3}[3\Delta] = \left(c_4^3 - c_6^2\right)$
- $[3\Delta]^2 = 3[3\Delta]$
- $3\alpha, 3\beta, 3[\alpha\Delta] = 0$
- α *or* β *times* $[3\Delta]$ *or* $\left[3\Delta^2\right]$ *is* 0
- $\alpha^2, [\alpha\Delta]^2, \beta^5 = 0$
- $\alpha\beta^2 = 0$
- α*,* β *or* $[\alpha\Delta]$ *times* c_4 *or* c_6 *is* 0

In particular, it is $\deg\Delta^3 = 3 \cdot 24 = 72$*-periodic.*

Remark 16. The computation of $\pi_*\mathrm{TMF}_{(2)}$ is even more involved. However, we remark that there is a horizontal vanishing line in the spectral sequence, and that it is $\deg\Delta^8 = 8 \cdot 24 = 192$-periodic. Similarly, the spectral sequence computing $\pi_*\mathrm{TMF}$ has a vanishing line, and it is $\deg\Delta^{24} = 24 \cdot 24 = 576$-periodic.

References

[Bau03] Tilman Bauer. "Computation of the homotopy of the spectrum tmf". In: (2003). eprint: `arXiv:math/0311328`.

[Beh19] Mark Behrens. "Topological modular and automorphic forms". In: (2019). eprint: `arXiv:1901.07990`.

[Goe09] Paul G. Goerss. "Topological modular forms (after Hopkins, Miller, and Lurie)". In: (2009). eprint: `arXiv:0910.5130`.

[Mat12] Akhil Mathew. "The homotopy groups of TMF". In: (2012). URL: http://math.mit.edu/~sglasman/tmfhomotopy.pdf.

[Mei17] Lennart Meier. "Elliptic Homology and Topological Modular Forms (Version 0.16)". In: (2017). URL: http://www.staff.science.uu.nl/~meier007/TMF-Lecture.pdf.

[MO17] Lennart Meier and Viktoriya Ozornova. "Moduli stack of elliptic curves". In: (2017). URL: http://www.staff.science.uu.nl/~meier007/Mell.pdf.

[Rez01] Charles Rezk. "Supplementary notes for math 512 (version 0.18)". In: (2001). URL: https://faculty.math.illinois.edu/~rezk/512-spr2001-notes.pdf.