Rational Homotopy

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In general, we have two classical algebraic invariants of a space: Its (co)homology and its homotopy groups. Taking cohomology $X \mapsto H^*X$ is easy to calculate, but losees a lot of information, and π_*X is difficult to compute. However, it turns out that all the complexity is in the torsion part: if we work rationaly, the story is different.

Definition 1. A space X is called rational if π_*X has the structure of a $\mathbb{Q}$-vector space.

Furthermore, for any space X, we can define its rationalization $X \to X_{\mathbb{Q}}$, a universal space with homotopy groups $\pi_*(X) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_*(X_{\mathbb{Q}})$. We'll give a precise definition later. For example, a model for the rational sphere $S^n_{\mathbb{Q}}$ is

$$S^n_{\mathbb{Q}} \simeq \left(\bigvee_{k\geq 1} S^n_k \right) \cup \left(\bigsqcup_{k\geq 2} D^n_k \right)$$

where the attaching maps $\partial D^n_{k+1} \to S^n_k \vee S^n_{k+1}$ are $1_{S^n_k} - (k+1)_{S^n_{k+1}}$, which represents the element $\frac{1}{k+1}$ in $S^n_{\mathbb{Q}}$. We define the cateogry $\mathrm{Top}^{\mathbb{Q}}$ as the category of simply connected rational topological spaces, and the functor $(-)_{\mathbb{Q}} : \mathrm{Top} \to \mathrm{Top}^{\mathbb{Q}}$ as the rationalization functor. Then the idea is that the category $\mathrm{Top}^{\mathbb{Q}}$ is simple, in the sense that the cohomological information is enough to recover the space and its homotopy groups.

The first hint is by what is called Hurevich mod $\mathcal{C}$.

Definition 2. A subcategory $\mathcal{C} \subset \mathcal{A}b$ is called a Serre class if for any short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

$M \in \mathcal{C}$ iff $M', M'' \in \mathcal{C}$, and $\mathcal{C}$ is closed under tensor product and $\mathrm{Tor}^{\mathbb{Z}}_1(-,-)$.

Example 3. The following are examples of Serre classes:

1. Finite abelian groups.

2. Finitely generated abelian groups.

3. Torsion abelian groups

The last example is the one important for us.

Fact 4. *For any pair of simply connected spaces* (X,Y), $\pi_k(X,Y)\in\mathcal{C}\forall k<n$ *iff* $H_n(X,Y)\in\mathcal{C}\forall k<n$.

Definition 5. A morphism $f:A\to B$ between abelian groups is called $\mathcal{C}$-monomorphism (epimorphism) if $\ker f$ ($\operatorname{coker} f$) belongs to $\mathcal{C}$. f is $\mathcal{C}$-isomorphism.

Using this definitions, we can state two of basic theorems of rational homotopy theory, stated originaly by Serre(?):

Theorem 6. *(Hurewicz Theorem mod* $\mathcal{C}$*) Let* $\mathcal{C}$ *be a Serre' class of abelian groups, and let* X *be a simply connected space. Suppose* $H_k(X)\in\mathcal{C}$ *for all* $k<n$ *(or equivalently* $\pi_k(X)$*). Then there is an exact sequence:*

$$K\to\pi_nX\to H_nX\to C\to 0$$

such that $K,C\in\mathcal{C}$*. In particular,* $\pi_nX\to H_nX$ *is a* $\mathcal{C}$*-isomorphism.*

Proof. Let $\{X_{\le n}\}$ be the Postnikov tower of X (that is, a sequence of spaces $X_{\le n}\to X_{\le n-1}\to\cdots$ such that $X\simeq\underleftarrow{\lim}X_{\le n}$, $\pi_{>n}X_{\le n}=0$ and $\pi_{\le n}X\xrightarrow{\sim}\pi_{\le n}X_{\le n}$). Then using the exact sequences of the pair (X_{n-1},X), together with the standard (and relative) Hurewicz homomorphisms:

$$\begin{array}{ccccccccc}
0=\pi_{n+1}(X_{n-1}) & \longrightarrow & \pi_{n+1}(X_{n-1},X) & \xrightarrow{\simeq} & \pi_n(X_n) & \longrightarrow & \pi_n(X_{n-1})=0 & \longrightarrow & \pi_n(X_{n-1},X)\\
\downarrow & & \simeq\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\mathcal{C}\ni H_{n+1}(X_{n-1}) & \longrightarrow & H_{n+1}(X_{n-1},X) & \longrightarrow & H_n(X_n) & \longrightarrow & H_n(X_{n-1})\in\mathcal{C} & \longrightarrow & H_n(X_{n-1},X)
\end{array}$$

□

Theorem 7. *(Whitehead mod* $\mathcal{C}$*) Let* $\mathcal{C}$ *be a Serre class,* $f:X\to Y$ *a map between simply connected spaces. Then the following are equivalent:*

1. $\pi_{\le n}(f)$ *is a* $\mathcal{C}$*-isomorphism and* $\pi_{n+1}(f)$ *is a* $\mathcal{C}$*-epimorphism.*
2. $H_{\le n}(f)$ *is a* $\mathcal{C}$*-isomorphism and* $H_{n+1}(f)$ *is a* $\mathcal{C}$*-epimorphism.*

Proof. Using the long exact sequences for the pair (Y,X) we see that condition 1 is equivalent to $\pi_k(Y,X)\in\mathcal{C}$ while 2 is equivalent to $H_k(Y,X)\in\mathcal{C}$. □

Combining these two theorems, we conclude:

Corollary 8. *For a map* $f:X\to Y$ *between simply connected spaces,*π_*f *is an equivalence iff* $H_*(f,\mathbb{Q})$ *is an isomorphism iff* $H^*(f,\mathbb{Q})$ *is an equivalence.*

So we see that H^* remembers some of the information about equivalences (if a map comes from a topological map then it remembers the information about equivalences) and we may ask what is the missing infirmation and whether or not we can encode it using a structure or some modification to the cohomology groups.

So only the information about H^* is not enough, even if we remember the ring structure. However, if we remmeber the structure of the chain complex itself, a chain with a differential and the cup product, then the answer will be positive. However, the problem is that $C^*(X;\mathbb{Q})$ with the cup product is not commutative and associative on the nose.

The calssical solution to this problem was to replace $C^*(X;\mathbb{Q})$ with a quasi-isomorphic chain complex that has a strict cdga structure: Sullivan defined such a model using local differential forms: for every singular simplex $\Delta^n \to X$ we can assosiate the group of differential forms on Δ^n with some compatibility between a form on a simplex and forms on its boundary, and together with the local differential of $\Omega_\bullet(\Delta^n)$, this will have the structure of a cdga.

Precisely, for a space X we have the presheaf $X_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Set}$ of the singular simplices, and the presheaf

$$\Omega_\bullet : \Delta^{\mathrm{op}} \to \mathrm{cdga}_{\mathbb{Q}} \xrightarrow{\text{fortegful}} \mathrm{Set}$$

. Then the differential forms on X will be natural transformations $X_\bullet \to \Omega_\bullet$. One can show that the set of natural transformations has a structure of a cdga, by applying the operations pointwise (or by Kan extension). Thus we obtain a functor

$$\left(\mathrm{Top}^{\mathbb{Q}}\right)^{\mathrm{op}} \to \mathrm{cdga}_{\mathbb{Q}}$$
$$X \mapsto \mathrm{Hom}_{\mathrm{sSet}}(X_\bullet, \Omega_\bullet)$$

This cdga is quasi-isomorphic to the singular chain, and Sullivan proved that this functor is fully faithful, and we have a simple characterisation for its essentual image.

However, there is a way to avoid the usage of differential forms, and use the singular chain itself: $C^*(X;\mathbb{Q})$ is indeed not strictly commutative, but it has the structure of an $\mathbb{E}_\infty$-ring. More precisely, we can use the following.

Definition 9. Let $H\mathbb{Q} \in \mathrm{Sp}$ be the spectrum representing rational cohomology. This is an $\mathbb{E}_\infty$-ring, so we can define the category $\mathrm{Mod}_{H\mathbb{Q}}$ of module spectra over $\mathbb{Q}$.

The important difference of rational homology from any other homology theories is the following observation:

Definition 10. For any spectrum E, we have the notion of E-acyclic spectra - Y s.t. $E \otimes Y \simeq *$, and E-local spectra which are those X s.t. for any E-acyclic Y and any $f : Y \to X$, f is nullhomotopic. Finally, a map $f : X \to Y$ is E-equivalence if $f \otimes E$ is an equivalence. A fundamental concept in stable homotopy theory is the notion of Bousfield localization: For any spectrum E there is a localization functor $L_E : \mathrm{Sp} \to \mathrm{Sp}_E$ s.t. $L_E(X)$ is E-local and $X \to L_E(X)$ is E-equivalence. If E is a ring, for example if $E = HR$ for some ordinary ring, then any E-module M is E-local, so $\mathrm{Mod}_E \subset \mathrm{Sp}_E$.

The spectrum $H\mathbb{Q}$ has two spetial properties: One is that $H\mathbb{Q} \simeq L_{H\mathbb{Q}}\mathbb{S}$, and the other is that this spectrum is "smashing", that is $L_{H\mathbb{Q}}(X)$ is given by $X \mapsto L_{H\mathbb{Q}}\mathbb{S} \otimes X$. Combining these two observations, we obtain that $H\mathbb{Q}$ localization is given by $X \mapsto H\mathbb{Q} \otimes X$. In particular, since L_E is an equivalence for E-local spectra, we obtaing that any $H\mathbb{Q}$-local spectra X is also an $H\mathbb{Q}$-module by $L_{H\mathbb{Q}}^{-1} : L_{H\mathbb{Q}}X \simeq X \otimes H\mathbb{Q} \to X$, so $\mathrm{Mod}_E \supset \mathrm{Sp}_E$ and we get:

Corollary 11. $\mathrm{Sp}_{H\mathbb{Q}} \simeq \mathrm{Mod}_{H\mathbb{Q}}$.

Now we can reformulate the idea of rational homotopy in the following way: For any ∞-category $\mathcal{C}$ and a set of morphisms W we can define the localization of $\mathcal{C}$ WRT W, denoted by $L_W : \mathcal{C} \to \mathcal{C}\left[W^{-1}\right]$, which is the universal category such that all the morphisms in W are invertible.

Given a ring R, we have two notions of R-local homotopy theory:

1. The first is the localization of $\mathcal{S}_{\geq 1}$ WRT $\pi_* \otimes R$ equivalences
2. The second is the localization Sp_{HR}. Since any connected spectrum is a commutative monoid in $\mathcal{S}$, we have a forgetful functor $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$, and this functor admits a left adjoint, called Σ_+^∞. So using this functor, we can also define the localization of spaces WRT HR-local spectra. That is, take the composition $\Sigma_{+,\mathbb{Q}}^\infty : \mathcal{S}_{\geq 1} \xrightarrow{\Sigma_+^\infty} \mathrm{Sp} \xrightarrow{L_{H\mathbb{Q}}} \mathrm{Sp}_{H\mathbb{Q}} \simeq \mathrm{Mod}_{H\mathbb{Q}}$. Since any space admits a diagonal map $\Delta : X \to X \times X$, this functor factors through

$$\begin{array}{ccc} \mathcal{S}_{\geq 1} & \xrightarrow{(-)\otimes H\mathbb{Q}} & \mathrm{coCAlg}\left(\mathrm{Mod}_{H\mathbb{Q}}\right) \\ & \searrow & \downarrow \\ & & \mathrm{Mod}_{H\mathbb{Q}} \end{array}$$

   So we can also define the localization WRT $(-) \otimes H\mathbb{Q}$-equivalences.

The second localization is localization WRT $H_*(-, \mathbb{Q})$ equivalences, so Serre's theorem is then that these two notions are equivalent, and we define:

Definition 12. $\mathcal{S}^{\mathbb{Q}}$ is the localization of $\mathcal{S}_{\geq 1}$ using any of the equivalent localizations above.

Serre's theorem also tells us that we can also take $H^*(-, \mathbb{Q})$ instead of $H_*(-, \mathbb{Q})$, i.e. that the functor

$$\left[\Sigma_+^\infty(-), H\mathbb{Q}\right] : \left(\mathcal{S}^{\mathbb{Q}}\right)^{\mathrm{op}} \to \mathrm{CAlg}\left(\mathrm{Mod}_{H\mathbb{Q}}\right) := \mathrm{CAlg}_{\mathbb{Q}}$$

is conservative. The question of rational homotopy theory is then whether or not this functor is also an embedding, and what is its essential image.

Remark 13. Even though the category $\mathrm{CAlg}_{\mathbb{Q}}$ seems complicated and non-algebraic, its actually equivalent to the catgory of $\mathrm{cdga}_{\mathbb{Q}}$, by

$$\left[\Sigma_+^\infty(-), H\mathbb{Q}\right] \to C^*(-; \mathbb{Q})$$

so we indeed recover the classical rational homotopy theory. Therefore, for now on we will identify between the chain complex $\mathbb{Q}$ concentrated in degree 0 and the spectrum $H\mathbb{Q}$, and the functor $C^*(-;\mathbb{Q})$ with $\left[\Sigma^\infty_+(-), H\mathbb{Q}\right]$.

In general this functor is not fully faithful, due to finitness problems. However, if we restrict to finite-type spaces, we will get a fully faithful functor, with a concrete characterisation of its essential image:

Theorem 14. *Let $\mathcal{S}^{\mathbb{Q}}_{\mathrm{ft}}$ be the category of rational spaces with $\pi_n X$ finite dimensional $\mathbb{Q}$-vector spaces for $n \geq 2$. Then the functor*

$$C^*(-;\mathbb{Q}) : \left(\mathcal{S}^{\mathbb{Q}}_{\mathrm{ft}}\right)^{\mathrm{op}} \to \mathrm{CAlg}_{\mathbb{Q}}$$

is fully faithful, and its essential image is algebras A with the following properties:

1. *$\pi_i A$ is finite dimensional $\mathbb{Q}$-vector spaces for $i < -2$*
2. *$\pi_{-1} A = 0$*
3. *$\pi_0 A \simeq \mathbb{Q}$*
4. *$\pi_{>0} A = 0$.*

The functor $C^*(-;\mathbb{Q})$ admits a right adjoint $A \mapsto \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{Q}}}(A, \mathbb{Q})$. The unit map is

$$\begin{aligned} \mathrm{eval} : X &\to \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{Q}}}\left(C^*(X;\mathbb{Q}), \mathbb{Q}\right) \\ x &\mapsto \mathrm{eval}_x \end{aligned}$$

and the counit map

$$\begin{aligned} A &\to C^*\left(\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{Q}}}(A, \mathbb{Q}); \mathbb{Q}\right) \simeq \mathrm{Map}\left(H\mathbb{Q}^A, H\mathbb{Q}\right) \\ a &\mapsto \mathrm{eval}_a \end{aligned}$$

In order to show that $C^*(-;\mathbb{Q})$ is fully faithful, we have to show that the unit map is an equivalence. Before proving the theorem, we will need some calculations.

Lemma 15. *$C^*(K(\mathbb{Q}, n); \mathbb{Q})$ is the free commutative algebra on the generator $\mathbb{Q}[-n]$, i.e. an exterior algebra $\Lambda^*(\mathbb{Q}[-n])$ for n odd and a polynomial algebra $\mathbb{Q}[x]$ with $|x| = n$ for n even. In general, for a finite dimensional $\mathbb{Q}$-vector space V, $C^*(K(V, n); \mathbb{Q}) \simeq \mathrm{Free}(V[-n])$.*

Proof. First we show that $\pi_m(C^*(K(\mathbb{Q}, n))) = H^*(K(\mathbb{Q}, n)) \simeq \pi_m \mathrm{Free}(\mathbb{Q}[-n])$. We prove this by induction on n: For the case $n = 1$, the map $\mathbb{Z} \to \mathbb{Q}$ induces a map $B\mathbb{Z} \to B\mathbb{Q}$ which is an isomorphism on rational homotopy, hence an isomorphism on rational cohomology, and

$$H^m(B\mathbb{Z};\mathbb{Q}) \simeq H^m\left(S^1;\mathbb{Q}\right) = \begin{cases} \mathbb{Q} & m = 0, 1 \\ 0 & \text{otherwise} \end{cases} = \pi_m(\mathbb{Q} \oplus \mathbb{Q}[-1]) = \pi_m(\Lambda^* \mathbb{Q}[-1])$$

since for $m \geq 2$ $\Lambda^m \mathbb{Q}[-1]$ is trivial: Its the quotient of $\mathbb{Q} \simeq \mathbb{Q} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} \mathbb{Q}$ by the action of Σ_m, which acts by multiplication by the sign of the permutation, i.e. $x = -x$.

Now assume the claim holds for $n-1$. We have the path-loop fibration

$$\begin{array}{ccccc} \Omega K(\mathbb{Q}, n) & \longrightarrow & PK(\mathbb{Q}, n) & \longrightarrow & K(\mathbb{Q}, n) \\ \simeq \downarrow & & \simeq \downarrow & & \\ K(\mathbb{Q}, n-1) & & \star & & \end{array}$$

We'll prove the case of n even, so by hypothesis, $H^q(K(\mathbb{Q}, n-1))$ is $\mathbb{Q}$ for $q = 0, n-1$ and zero otherwise, and we can use the Serre spectral sequence for this fibration to compute $H^* K(\mathbb{Q}, n)$: The E_2 page is

$$E_2^{p,q} = H^p(K(\mathbb{Q}, n); H^q(K(\mathbb{Q}, n-1))) \simeq H^p(K(\mathbb{Q}, n); \mathbb{Q}) \otimes H^p(K(\mathbb{Q}, n); \mathbb{Q}) \Rightarrow H^{p+q}(*)$$

and $E_2^{p,q} = 0$ for $q \neq 0, n-1$, so the only non-zero differential is $d_n : E_n^{p,n-1} \to E_n^{p+n,0}$. Since $E_\infty^{p,q} = 0$, d_n is an isomorphism. Take a generator $x \in E_n^{0,n-1} \simeq H^{n-1}(K(\mathbb{Q}, n-1))$, and let $y = d_n(x) \in E_n^{n,0} \simeq H^n(K(\mathbb{Q}, n))$. Then y generates $H^n(K(\mathbb{Q}, n))$, and xy generates $E_n^{n,n-1} \simeq H^n(K(\mathbb{Q}, n)) \otimes H^{n-1}(K(\mathbb{Q}, n-1))$. Then again, $d_n(xy)$ generate $E_n^{2n,0} \simeq H^{2n}(K(\mathbb{Q}, n))$, and since d_n is a derivation $d_n(xy) = d_n(x)y + x d_n(y) = y^2$. We continue this way to show that y^k generates $H^{kn}(K(\mathbb{Q}, n))$, i.e. $H^{kn}(K(\mathbb{Q}, n)) \simeq \mathbb{Q}[y]$.

Thus, $\pi_* C^*(K(\mathbb{Q}, n); \mathbb{Q}) \simeq \pi_* \operatorname{Free}(\mathbb{Q}[-n])$. Choose an element $\alpha \in \pi_* C^*(K(\mathbb{Q}, n); \mathbb{Q})$ that maps to a generator of $\pi_* \operatorname{Free}(\mathbb{Q}[-n])$. Then α factors as a map $\mathbb{Q}[-n] \to C^*(K(\mathbb{Q}, n); \mathbb{Q})$, and thus extends to a map $\operatorname{Free}(\mathbb{Q}[-n]) \to C^*(K(\mathbb{Q}, n); \mathbb{Q})$. This map is an isomorphism on homotopy, and thus an isomorphism. □

As a consequence of this computation, we can actually compute now the rational homotopy groups of spheres!

Claim 16. For n odd, $\pi_k(S^n_{\mathbb{Q}}) \simeq \begin{cases} \mathbb{Q} & k = n \\ 0 & \text{otherwise} \end{cases}$, and for n even, $\pi_k(S^n_{\mathbb{Q}}) \simeq \begin{cases} \mathbb{Q} & k = n, 2n-1 \\ 0 & \text{otherwise} \end{cases}$. In particular, $\pi_k(S^n)$ is finite for $k \neq n$ if n is odd and for $k \neq n, 2n-1$ if n is even.

Proof. For any n, $H^n(-, \mathbb{Q})$ is represented by $K(\mathbb{Q}, n)$. In particular $\mathbb{Q} \simeq H^n(S^n; \mathbb{Q}) \simeq [S^n, K(\mathbb{Q}, n)] = \pi_n(K(\mathbb{Q}, n))$. Choose some $f : S^n \to K(\mathbb{Q}, n)$ representing a non-zero element (hence, a generator). By Hurewicz, f is an isomorphism on $H^n(K(\mathbb{Q}, n); \mathbb{Q}) \to H^n(S^n; \mathbb{Q})$. So, for n odd we are done: f is an isomorphism on cohomology, hence on homotopy, so $\pi_* S^n_{\mathbb{Q}} \simeq \pi_* K(\mathbb{Q}, n)$.

For n even, we can write $H^*(K(\mathbb{Q}, n); \mathbb{Q}) \simeq \mathbb{Q}[x]$ for $x \in H^n(K(\mathbb{Q}, n); \mathbb{Q}) \simeq \mathbb{Q}$ a generator. Then $x^2 \in H^{2n}(K(\mathbb{Q}, n); \mathbb{Q}) \simeq [K(\mathbb{Q}, n), K(\mathbb{Q}, 2n)]$. Chose a representative $g : K(\mathbb{Q}, n) \to K(\mathbb{Q}, 2n)$ for x^2, and let $F \to K(\mathbb{Q}, n)$ be its fiber. Since $\pi_n(K(\mathbb{Q},2n)) = 0$, $f : S^n \to K(\mathbb{Q},n)$ factors through the fiber

$$\begin{array}{ccccc} F & \longrightarrow & K(\mathbb{Q},n) & \xrightarrow{g} & K(\mathbb{Q},2n) \\ & \nwarrow^{h} & \uparrow f & & \\ & & S^n & & \end{array}$$

Since $\pi_n(f)$ is an isomorphism so is $\pi_n(h)$, and thus by Hurewicz on $H^n(h)$. Its possible to show that the induced map on cohomology is a cofiber sequence, i.e.

$$H^*(F;\mathbb{Q}) \simeq \mathbb{Q}[x]/\left(x^2\right)$$

so h is an isomorphism on cohomology, hence on $\pi_* \otimes \mathbb{Q}$. In particular, $S^n_{\mathbb{Q}} \simeq F$, and using the long exact sequence in homotopy we obtain $\pi_k\left(S^n_{\mathbb{Q}}\right) \simeq \begin{cases} \mathbb{Q} & k = n, 2n-1 \\ 0 & \text{otherwise} \end{cases}$.

□

Now we go back to prove 14:

Claim 17. The unit map eval : $X \to \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{Q}}}(C^*(X;\mathbb{Q}), \mathbb{Q})$ is an isomorphism.

Proof. We use the following sequence of arguments:

First, we reduce to the case where X is n-truncated for some n. This is done by using $X_{\leq n}$, the n^{th} Postnikov space, i.e. a space with $X \to X_{\leq n}$ is an isomorphism on $\pi_{\leq n}$ and $\pi_{>n}(X_{\leq n}) = 0$. Since $X \simeq \varprojlim X_n$, and X is simply connected, $C^*(X;\mathbb{Q}) \simeq \varinjlim C^*(X_{\leq n};\mathbb{Q})$. Thus its suffices to show the claim for $X_{\leq n}$, so we can use an inductive argument on n, where the base case $n = 1$ is obvious since X is simply connected.

Next, we use the fact that at least for X simply connected of finite type, the fiber sequence

$$\begin{array}{ccc} K(\pi_n X, n) & & \\ \downarrow & & \\ X_{\leq n} & \longrightarrow & X_{\leq n-1} \end{array}$$

is classifies by a pullback square

$$\begin{array}{ccc} X_{\leq n} & \longrightarrow & \star \\ \downarrow & & \downarrow \\ X_{\leq n-1} & \longrightarrow & BK(\pi_n X, n) \simeq K(\pi_n X, n+1) \end{array}$$

(in general it always classified by $B\,\mathrm{Aut}(K(\pi_n X, n)) \simeq \mathrm{Aut}(\pi_n X) \rtimes K(\pi_n X, n+1)$). Then we use the fact that at least for finite type simply connected spaces, the functor $C^*(-;\mathbb{Q})$ sends cofiber sequences to fiber sequences, i.e. we have a pushout diagram

$$\begin{array}{ccc} C^*(K(\pi_n X, n+1);\mathbb{Q}) & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ C^*(X_{\leq n-1};\mathbb{Q}) & \longrightarrow & C^*(X_{\leq n};\mathbb{Q}) \end{array}$$

Now by the inductive step $X_{\leq n-1} \simeq \mathrm{Map}_{\mathrm{Calg}_{\mathbb{Q}}}(C^*(X_{\leq n-1};\mathbb{Q}),\mathbb{Q})$, and by lemma **??**

$$\begin{aligned} \mathrm{Map}_{\mathrm{Calg}_{\mathbb{Q}}}(C^*(X_{\leq n-1},\mathbb{Q}),\mathbb{Q}) &\simeq \mathrm{Map}_{\mathrm{Calg}_{\mathbb{Q}}}(\mathrm{Free}(\mathbb{Q}[-n-1]),\mathbb{Q}) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Q}}}\left(\Sigma^{-n-1}H\mathbb{Q}, H\mathbb{Q}\right) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Q}}}\left(H\mathbb{Q}, \Sigma^{n+1}H\mathbb{Q}\right) \\ \{\star\} &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Q}}}\left(\mathbb{S}, \Sigma^{n+1}H\mathbb{Q}\right) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{\mathbb{Q}}}\left(\Sigma^{\infty}_{+}(*), \Sigma^{n+1}H\mathbb{Q}\right) \\ &\simeq \mathrm{Map}_{\mathcal{S}}\left(*, \Omega^{\infty}\Sigma^{n+1}H\mathbb{Q}\right) \\ &\simeq \Omega^{\infty}\Sigma^{n+1}H\mathbb{Q} \\ &\simeq K(\pi_n X, n+1) \end{aligned}$$

where $\star$ is since any map into an $H\mathbb{Q}$-local spectra factors through the localization, and $L_{H\mathbb{Q}}\mathbb{S} \simeq H\mathbb{Q}$. Thus this holds also for $X_{\leq n}$. ☐

Corollary 18. *For any $X, Y \in \mathcal{S}^{\mathbb{Q}}_{\mathrm{ft}}$,*

$$\mathrm{Map}_{\mathcal{S}^{\mathbb{Q}}}(Y, X) \simeq \mathrm{Map}_{\mathrm{Calg}_{\mathbb{Q}}}(C^*(X; H\mathbb{Q}), C^*(Y;\mathbb{Q}))$$

i.e. $C^*(-;\mathbb{Q})$ is fully-faithful.

We now want to describe the essential image, and show that its precisely algebras A that satisfies conditions $1-4$.

First, let A be in the image. Conditions 2-4 are obvious. In order to show that $C^n(X;\mathbb{Q})$ are finite dimensional we use again an inductive argument and the esplicit characterisation of the cofibers $X_{\leq n} \to X_{\leq n-1} \to K(\pi_n X, n+1)$. Its remains to prove that a cdga satisfying conditions 1-4 is equivalent to the cohomology of a rational space. Let $A \in \mathrm{CAlg}_{\mathbb{Q}}$. If A where indeed of the form $C^*(X;\mathbb{Q})$, then we could recover X as $\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{Q}}}(A,\mathbb{Q})$, and in particular $X \in \mathcal{S}^{\mathbb{Q}}_{\mathrm{ft}}$, but since we don't know yet that A is in the essential image of $C^*(-;\mathbb{Q})$, even if X is indeed a finite type rational space we still don't know its corresponding to A. The idea is then to look at $X = \mathrm{Map}(A,\mathbb{Q})$ as the rational point of the functor

$$\mathcal{X}_A = \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{Q}}}(A, -) : \mathrm{CAlg}^{\geq 0}_{\mathbb{Q}} \to \mathcal{S}$$

For a general field the restricted Yoneda

$$\mathcal{X}_{(-)} : \left(\mathrm{CAlg}_k^{\leq 0}\right)^{\mathrm{op}} \to \mathrm{Fun}\left(\mathrm{CAlg}_k^{\geq 0}, \mathcal{S}\right)$$

is not an embedding. However, for a field of characteristic zero this is indeed an embedding. The important properties of such functors are:

1. Since we are mapping from a coconnective algebra to connective algebras, all the information is in its values on discrete algebras (and its value on any connective algebra is its left Kan extension). i.e. the composition

$$\mathcal{X}_{(-)} : \left(\mathrm{CAlg}_{\mathbb{Q}}^{\leq 0}\right)^{\mathrm{op}} \to \mathrm{Fun}\left(\mathrm{CAlg}_{\mathbb{Q}}^{\geq 0}, \mathcal{S}\right) \to \mathrm{Fun}\left(\mathrm{CAlg}_{\mathbb{Q}}^{0}, \mathcal{S}\right)$$

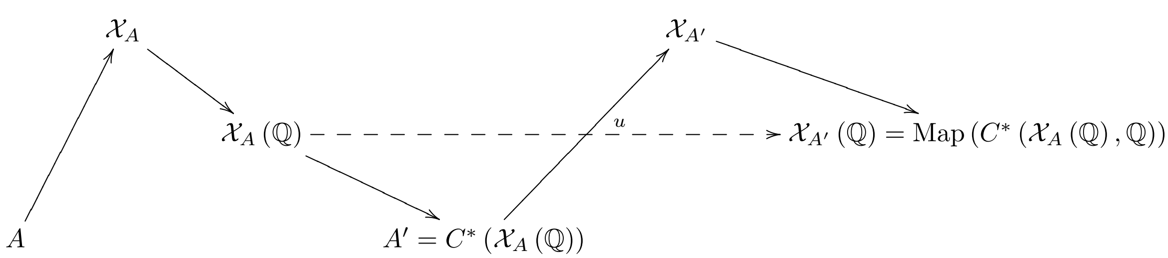
   is also an embedding.

2. If A is $-n$ truncated, then the values of $\mathcal{X}_A$ are n-connected.

3. For all i, the functor $R \mapsto \pi_i \mathcal{X}_A(R)$ restricted to $\mathrm{CAlg}_{\mathbb{Q}}^0$ is given by $R \mapsto R \otimes_{\mathbb{Q}} V$ for a finite-dimensional $\mathbb{Q}$ vector space V.

In particular, from property 2 and 3 we deduce that indeed the rational points are rational spaces of finite type $\mathcal{X}_A(\mathbb{Q}) \in \mathcal{S}_{\mathrm{ft}}^{\mathbb{Q}}$.

Now in order to prove that A is in the essential image of $C^*(-;\mathbb{Q})$, we use define $A' \coloneqq C^*(\mathcal{X}_A(\mathbb{Q});\mathbb{Q})$. Then we want to show that $A' \simeq A$. We have the following diagram:



and we wish to show that the composition $A \to A'$ is an equivalence. Note that u is the unit map of the adjunction $\left(\mathcal{S}_{\mathrm{ft}}^{\mathbb{Q}}\right)^{\mathrm{op}} \rightleftarrows \mathrm{CAlg}_{\mathbb{Q}}^{<1}$, hence an equivalence. Since $\pi_i \mathcal{X}_{(-)}(R)$ are $\mathbb{Q}$ are finite $\mathbb{Q}$-vector spaces for any connective R, this implies that $\mathcal{X}_A \to \mathcal{X}_{A'}$ is also an equivalence $\mathcal{X}_A \to \mathcal{X}_{A'}$ and hence $A \xrightarrow{\sim} A$.

This observation now gives us a hint about how to extend the embedding to all rational spaces, and not just finite type spaces: We define the subcategory $\mathrm{Fun}\left(\mathrm{CAlg}_{\mathbb{Q}}^{\geq 0}, \mathcal{S}\right) \supset \mathrm{RType}$ spanned by those functors satisfying properties 1-3, except that we don't demant V to be finite-dimensional in 3. Then its possible to show that taking rational points induces an equivalence:

$$(-)(\mathbb{Q}) : \mathrm{Fun}\left(\mathrm{CAlg}_{\mathbb{Q}}^{\geq 0}, \mathcal{S}\right) \supset \mathrm{RType} \to \mathcal{S}^{\mathbb{Q}}$$