Kan Seminar
∞-Categories and Spectra

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We want to do mathematics, where we replace sets by *spaces* (i.e. homotopy types). The main slogan of today is that we would like to remove the notion of equality, i.e. that things are not equal but rather we *specify* an *equivalence*, or a homotopy, between them.

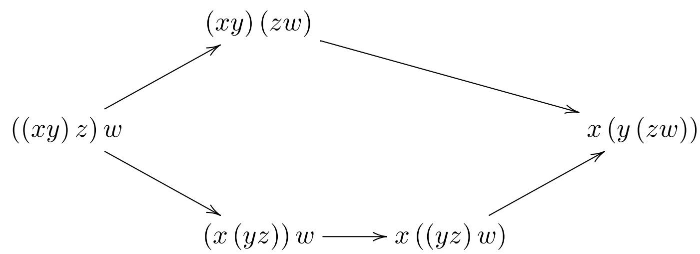
1 ∞-Monoids and ∞-Groups

First recall that a monoid is the same as a group but without an inverse. All groups are in particular monoids. Other typical examples are $\mathbb{N}$, and $\mathrm{End}(x) = \hom(x, x)$ in any category (with composition). So let's say that we want to have a monid in the world of homotopy.

A first idea (which will turn out to be badly behaved homotopically) of what might be a monoid in homotopy is a monoid object G in the category Top of topological spaces: an element $1 \in G$ and continous $\cdot : G \to G$, such that the composition is associative and unital. Examples include all usual monoid/groups (with discerete topology), S^1 (with multiplication by embedding it in $\mathbb{C}$), matrix groups such as $\mathrm{GL}_n(\mathbb{R}), \mathrm{GL}_n(\mathbb{C}), \mathrm{U}(n), \mathrm{SO}(n)$ with matrix multiplication (with the topology coming from the embedding in $M_n(\mathbb{F}) \cong \mathbb{F}^{n^2}$), $\mathrm{End}(X)$ in topological spaces (endowed with the compact-open topology) etc.

Now, note that for an ordinary monoid G, a set X, and a bijection of sets $G \cong X$, we can transport the monoid structure to X. Specifically, let $\psi : X \to G$ be the bijection, then $xy = \psi^{-1}(\psi x \cdot \psi y)$. Similarly, if G is a topological monoid, and X is another space, and we have a *homotopy equivalence* $G \cong X$, we would like to be able to transport the structure to X. This would work as above for a homeomorphism, but we want our notions to be homotopy invariant, but following the lines above we won't get the same kind of an object. Let's see what we get. Let $\varphi : G \rightleftarrows X : \psi$ be the homotopy inverse maps. We can still define $xy = \varphi(\psi x \cdot \psi y)$, giving a map $\cdot : X \times X \to X$. However, this map is not associative, namely $(xy)z = \varphi(\psi\varphi(\psi x \cdot \psi y) \cdot \psi z) \neq \varphi(\psi x \cdot \psi\varphi(\psi y \cdot \psi z)) =$

$x(yz)$, so we have two different points in $\hom\left(X^3, X\right)$. As we can easily see, if $\psi\varphi = \mathrm{id}_G$ (as is the case when the maps are inverses but not merely homotopy inverses) then they are equal. In our case they are not equal, but we do have a homotopy $\psi\varphi \rightsquigarrow \mathrm{id}_G$ (i.e. a map $G \times I \to G$). This gives us a path, or a homotopy, between the two ways to points in $\hom\left(X^3, X\right)$, i.e. $(xy)z \rightsquigarrow x(yz)$. This means that the multiplication on X is associative up to a specified homotopy in this sense. But wait, if we have 4 elements, there are 5 ways to put brackets on $xyzw$, and we have already chosen some ways to move between them:



As you can see, the paths we have chosen between the 5 points of X form a circle, that is we have a map $S^1 \to \hom\left(X^4, X\right)$. It turns out that using the homotopies $\psi\varphi \rightsquigarrow \mathrm{id}_G$ and $\varphi\psi \rightsquigarrow \mathrm{id}_X$ one can define a disk filling this circle, i.e. a homotopy witnessing the associativity of multiplication of 4 elements. One can continue this procedure and see that we have *higher coherencies* exhibiting different ways to multiply elements. One can axiomtize all this structure on X, which we call an ∞-monoid (also called $\mathbb{E}_1$-algebra, homotopically coherent monoid, or simply monoid). It turns out that this is the very good definition to work with, which indeed encompasses all the homotopically meaningful information. In fact, any ∞-monoid can be rigidified, i.e. it comes from a topological monoid.

In fact, if G is a topological group (and not merely a monoid), then we get that the map $X^2 \to X^2$ given by $(x, y) \mapsto (x, xy)$ is an equivalence, in which case we say that X is an ∞-group (sometimes called group-like).

An example of an ∞-group, which appears naturally, and not as something transported from a topological group, is a loop space. Let Y, y_0 be a pointed connected space. We know that $\pi_1(Y, y_0)$ is a group. The way the underlying set is defined is by considering all paths in Y based at y_0, and *identifying* homotopic paths. Identifying means making equal, but we wanted to get rid from equality and use spaces rather than sets. Therefore it makes sense to look at the space $X := \Omega Y = \left\{\gamma : S^1 \to Y \text{ based at } y_0\right\}$ of loops (endowed with the compact-open topology, in which "nearby loops" are "nearby"). By definition, a path in this space is a homotopy between loops (in the compact-open topology, continous $H : I \to \Omega Y$ corresponds to continous $I \times S^1 \to Y$). Therefore we conclude that the connected components $\pi_0 X = \pi_1(Y, y_0)$ form a group. In fact, it is easy to see that $\pi_n X = \pi_{n+1} Y$. Furthermore, we can define a multiplicative structure on X by path concatenation. Clearly this is not associative on the nose, but we can *choose* homotopies between paths $(\gamma * \delta) * \varepsilon$ to $\gamma * (\delta * \varepsilon)$. One can then see that we can fill the disk for 4 elements and so on. Thus this arranges to an ∞-group, and in fact

Theorem 1. *Let X be an ∞-group, then $X \cong \Omega Y$ for some (unique up to homotopy) pointed connected space Y.*

Corollary 2. *Let G be (discrete) group, then there exists a pointed connected space $\mathrm{B}G$ such that $G \cong \Omega \mathrm{B}G$. In particular $\pi_1 \mathrm{B}G = G$ and $\pi_{\geq 2} \mathrm{B}G = 0$.*

2 ∞-Categories

At this point we see that it is sensible to consider operations associative up to coherent homotopies. In particular, we would like to have a notion similar to categories in which structures like spaces (up to homotopy), ∞-groups, etc. live, and where we can manipulate them systematically. This is the notion of an ∞-category.

We first think back on categories. A category with one element x is the same data as a monoid, given by $\mathrm{End}(x)$. To get general categories then we need to have a collection of objects and morphisms between them, that can be composed when the target and source match. This can be though of "multi-object" monoids. Similarly, once we have an established notion of an ∞-monoid, by some modifications we get the notion of an ∞-category. In particular, an ∞-category has a space of objects, and between two objects we have a hom-*space*, with composition which associative *up to coherent homotopies* similarly to ∞-monoids, e.g. we have two ways to compose $\hom(x, y) \times \hom(y, z) \times \hom(z, w) \to \hom(x, w)$, and we have specified homotopy between them.

Given an ∞-category $\mathcal{C}$, one can get an ordinary category called the *homotopy category*, $\mathrm{h}\mathcal{C}$, whose objects are the (connected components) of $\mathcal{C}$, and $\hom_{\mathrm{h}\mathcal{C}}(x, y) = \pi_0 \hom(x, y)$.

This turns out to be a surprisingly good idea. Examples include all usual categories such as Set, Mon, Grp, Ab, etc. Furhtermore, spaces arrange into an ∞-category denoted $\mathcal{S}$ (where $\hom(X, Y)$ has the homotopy type of the compact-open topology). Moreover, ∞-monoids and ∞-groups arrange into ∞-categories Mon_∞ and Grp_∞.

Another example comes from spaces. Recall that for a pointed connected space Y, y_0 we defined $X = \Omega Y$ which is an ∞-monoid, a "one-object" ∞-category. Similarly, we can take a general (unpointed) space Y, and define the ∞-category whose space of objects is just Y, and $\hom(y, y')$ is the space of paths $y \rightsquigarrow y'$ (the case $y = y'$ recovers ΩY). Just like ΩY was an ∞-group, all the morphisms here are invertible (for each path take the reverse path), thus this ∞-category is in fact an ∞-*groupoid*. Clearly one can also recover the space Y from the

∞-groupoid, as the space of objects. So similarly to theorem which says that ∞-groups are loop spaces, we have the following

Theorem 3 (The Homotopy Hypothesis)**.** *An ∞-groupoid is the same thing as a space, i.e. there is an equivalence of ∞-categories $\mathcal{S} \cong \mathrm{Grpd}_\infty$.*

The theory of ∞-categories works very well, and has many similarities to category theory, with appropriate changes. For example, there are ∞-functors, adjunctions, ∞-co/limits (also called homotopy or derived co/limits), natural transformations etc. In this world, sets are replaced by spaces. As an example, the Yoneda lemma reads

Lemma 4 (Yoneda)**.** *Let $\mathcal{C}$ be an ∞-category, then the functor $\mathcal{C} \to \mathrm{Fun}\left(\mathcal{C}^{\mathrm{op}}, \mathcal{S}\right)$ (which sends $X \in \mathcal{C}$ to $h_X(Y) = \hom(X, Y)$) is fully faithful.*

2.1 ∞-Co/Limits

As we said, there is a notion of ∞-co/limits, and they have a similar universal property to that of usual co/limits, but in an ∞-category. To give an example, and to show that it plays nicely with homotopy theory, we consider the following. Let $f : X \to Y$ be a map of spaces. The cofiber is the pushout/colimit diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \mathrm{cofib}(f) \end{array}$$

i.e. $\mathrm{cofib}(f) = Y/f(X)$. Now, take $f : S^n \to \mathrm{pt}$, in this case we clearly get $\mathrm{cofib}(f) = \mathrm{pt}$. However, if we replace f with the equivalent map $f : S^n \to D^{n+1}$ which includes the boundary, we get $\mathrm{cofib}(f) = S^{n+1}$ (as we collapse the boundary of D^{n+1} to a point). We see that by replacing the diagram by a homotopy equivalent one, we get a colimit which is not homotopy equivalent (since S^{n+1} is not contractible).

The ∞-cofiber, or homotopy cofiber, which we denote for short by Cf is defined as the ∞-colimit. That is, it has the universal property that giving a map $C(f) \to Z$ is the same as giving a map $Y \to Z$, together with a homotopy from the image of $f(X)$ to a point in Z. You may have seen a construction that encodes just this, i.e. computes the homotopy cofiber, namely the mapping cone $X \times I \coprod Y/\sim$ where we crush $(x, 0)$ to a single point, and glue $(x, 1) \sim f(x)$. One can intuitively see that the "cone part" exactly gives the desire homotopy of the image of $f(X)$ to a point. Here it is easy to see that $C(S^n \to \mathrm{pt})$ and $C\left(S^n \to D^{n+1}\right)$ are both homotopy equivalent to S^{n+1}.

To connect this to other notions, lets work in the pointed category, in which we should also identify $\{x_0\} \times I$ to a point. Take the unique map $f : X \to \mathrm{pt}$, then we get that $Cf = \Sigma X$ is the reduced suspension (taking the unpointed version would give the non-reduced suspension). Unpacking the definition, this we have a homotopy pushout

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & \Sigma X \end{array}$$

thus the suspension has a very natural description. We also have a dual story of the homotopy fiber of a map. Again, working in pointed spaces and taking the unique map $\mathrm{pt} \to X$, one sees that we get a homotopy pullback diagram

$$\begin{array}{ccc} \Omega X & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & X \end{array}$$

Hopefully these examples indicate how useful are ∞-co/limits, and give you an idea on how to think about them.

3 Higher Algebra and Spectra

We have seen a (hint of the) definition of ∞-monoids and ∞-groups. This is a starting point for higher algebra. Just as ordinary algebra studies things like abelian groups, rings, etc., higher algebra studies the homotopy coherent versions of such structures. We shall now discuss (one of the versions of) ∞-abelian groups, i.e. commutative ∞-groups.

We have seen before that an ∞-group X_0 was the same as a loop space of a pointed connected space, $X_0 \cong \Omega X_1$. This was reasonable, as on ΩX_1 we have the concatenation of paths, which clasically gives the group structure on $\pi_1 X_1$. Now, we recall that that $\pi_{\geq 2}$ is always abelian. Taking this hint, we consider double loop spaces. That is, assume that X_1 itself is a loop space of a pointed 1-connected space $X_1 \cong \Omega X_2$. Then $X_0 \cong \Omega^2 X_2$, thus $\pi_n X_0 = \pi_{n+2} X_2$ so all of its homotopy groups are abelian. We recall that the way we prove that $\pi_{\geq 2}$ is abelian is by Eckmann-Hilton, one shows that the two ways to concatenate (put the squares side-by-side or on top of each other) distribute. We can define all the concatenations and the homotopies between them on $X_0 \cong \Omega^2 X_2$ itself. This indeed gives X_0 some coherent commutativity structure, i.e. different homotopies between xy and yx, and between three elements and so on. However, now one can ask, maybe X_2 is deloopable once more, i.e. $X_2 \cong \Omega X_3$. Then we get even more ways to concatenate, which gives even more homotopy coherence. If we can continue in this fashion infinitely many times, i.e. deloop X_0 more and more, then we obtain as much commutativity as possible on X_0. So a commutative ∞-group X is the same as a series of pointed n-connected spaces

$X_0, X_1, X_2, \dots$ together with equivalence $X_n \cong \Omega X_{n+1}$. We think of X_0 as the underlying pointed space, and the X_n's encode the coherent multiplication. These assemble into a category, called *connective spectra* $\mathrm{Sp}_{\geq 0}$. We have a forgetful (∞-)functor $\Omega^\infty : \mathrm{Sp}_{\geq 0} \to \mathcal{S}_*$ taking the underlying space $X \mapsto X_0$. The homotopy groups of X are defined to be the homotopy groups of X_0, i.e. $\pi_m X = \pi_m X_0 = \pi_m \Omega^n X_n = \pi_{m+n} X_n$ (which are all abelian, since we can take $n \geq 2$).

A simple example is $X = 0$ where all the spaces are just points.

We have seen that a discrete group G is always deloopable, i.e. $G \cong \Omega \mathrm{B}G$. Now if $\mathrm{B}G$ is deloopable itself, then $G = \pi_2 \mathrm{B}^2 G$ is abelian, and this is an if and only if, for an abelian group A, $\mathrm{B}A$ is deloopable to $\mathrm{B}^2 A$. One can keep on going and deloop it to $\mathrm{B}^n A$. We note that $\pi_m \mathrm{B}^n A = \begin{cases} A & m = n \\ 0 \end{cases}$ and $\pi_m \mathrm{H}A = \pi_m A = \begin{cases} A & m = 0 \\ 0 \end{cases}$. These spaces are known to represent cohomology, i.e. $H^n(X; A) = [X, \mathrm{B}^n A] = \pi_0 \hom(X, \mathrm{B}^n A)$. This shouldn't be too surprising, as a strictly commutative muliplication should of course commute in the homotopical sense. All of these then arrange into a connective spectrum, denoted $\mathrm{H}A$. We see that there is an (∞-)functor $\mathrm{H} : \mathrm{Ab} \to \mathrm{Sp}_{\geq 0}$, furthemore

Theorem 5. *The ∞-functor $\mathrm{H} : \mathrm{Ab} \to \mathrm{Sp}_{\geq 0}$ is fully faithful.*

This all arranges into a commutative square of ∞-categories:

$$\begin{array}{ccc} \mathrm{Ab} & \overset{\mathrm{H}}{\hookrightarrow} & \mathrm{Sp}_{\geq 0} \\ \downarrow & & \downarrow \Omega^\infty \\ \mathrm{Set}_* & \hookrightarrow & \mathcal{S}_* \end{array}$$

In ordinary algebra, there is a left adjoint to $\mathrm{Ab} \to \mathrm{Set}_*$, given by the free abelian group (identifying the base point with 0) $X \mapsto \mathbb{Z}[X]$. Similarly, there is a left adjoint (in the ∞-categorical sense) to Ω^∞, usually denoted by $\Sigma^\infty X$. This doesn't have a simple description as an infinite loop space, but the machinery of ∞-categories gives it for free.

There is another functor $\Sigma : \mathrm{Sp}_{\geq 0} \to \mathrm{Sp}_{\geq 0}$ which takes $X = (X_0, X_1, X_2, \dots)$ to $\Sigma X = (X_1, X_2, X_3, \dots)$, and one can check that it is the homotopy pushout with 0. Since $X_0 \cong \Omega X_1$, it is expected that this has an inverse from the left, and indeed, the adjoint $\Omega : \mathrm{Sp}_{\geq 0} \to \mathrm{Sp}_{\geq 0}$ is the left inverse. Furthermore, it is the homotopy pullback with 0. We also see that $\pi_n \Sigma X = \pi_n X_1 = \pi_{n+1} X_0 = \pi_{n+1} X$, so it has homotopy groups only from 1 and above. The full subcategory of these is denoted $\mathrm{Sp}_{\geq 1}$, and upon restriction of the adjunction we get an equivalence $\Sigma : \mathrm{Sp}_{\geq 0} \rightleftarrows \mathrm{Sp}_{\geq 1} : \Omega$. This should remind you positively-graded chain complexes, where we can shift in both directions, but only one of them is invertible. We can also formally invert the shift giving rise to $\mathbb{Z}$-graded chain complexes, and here similarly giving rise to the category of all spectra Sp, where Ω and Σ are inverse to each other and shift the homotopy groups. In particular, there are spectra with *negative homotopy groups*, e.g. $\pi_{-1}\Omega \mathrm{H}A = \pi_0 \mathrm{H}A = A$.

The world of spectra behaves like abelian groups in more ways. For example, there is a tensor product $X \otimes Y$, making it symmetric monoidal, with a unit denote by $\mathbb{S}$. Furthermore, just like the hom-set between abelian groups can be promoted to an abelian group, the hom-space between spectra can be upgraded to a spectrum (the adjoint to the tensor product), the internal $\underline{\hom}$. One can then check by playing with adjunction that $\pi_{-n}\underline{\hom}(X,Y) = \pi_0\underline{\hom}(X,\Sigma^n Y)$. Now, let X a space, we get:

$$\begin{aligned} \pi_{-n}\underline{\hom}(\Sigma^\infty X, \mathrm{H}A) &= \pi_0\underline{\hom}(\Sigma^\infty X, \Sigma^n \mathrm{H}A) \\ &= \pi_0\underline{\hom}(X, \Omega^\infty\Sigma^n \mathrm{H}A) \\ &= [X, \Omega^\infty \Sigma^n \mathrm{H}A] \\ &= [X, \mathrm{B}^n A] \\ &= H^n(X;A) \end{aligned}$$

thus $\underline{\hom}(\Sigma^\infty X, \mathrm{H}A)$ encodes the ordinary cohomology of X. Replacing $\mathrm{H}A$ by another spectrum E we get other cohomology theories.

One may wonder than what are $\mathrm{H}A \otimes \mathrm{H}B$ and $\underline{\hom}(\mathrm{H}A, \mathrm{H}B)$. On the π_0 these recover $A \otimes B$ and $\hom(A,B)$, however these objects are far richer, and are very much related to our next talk on the Steenrod algebra.

We can continue and develop the world of higher algebra. As a hint, if R is an abelian group, a ring structure on it includes in particular a map $R \otimes R \to R$. Following this idea in the world of spectra gives rise to ring spectra.