Higher Semiadditive Algebraic K-Theory and Redshift

joint with Tomer Schlank, arXiv:2111.10203

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https://shaybm.com/pdfs/notes/2022-02-04-upenn-semiadditive-k.pdf

Plan

- Chromatic homotopy theory
- Algebraic K-theory and redshift
- Higher semiadditivity, height and redshift
- Higher semiadditive K-theory and redshift
- Closing remarks

Homotopy Theory

Slogan

Redo math with *spaces* and *paths* instead of sets and equalities

- ∞-category – space of morphisms, composition associative and unital up to specified homotopies
- $\mathrm{Mon}(\mathcal{S})$ – space with multiplication associative and unital up to specified homotopies
- $\mathrm{CMon}(\mathcal{S})$ – ... commutative
- $\mathrm{CMon}^{\mathrm{gl}}(\mathcal{S}) = \mathrm{Sp}_{\geq 0}$ – ... group-like
- Sp – stabilization
  - $(\mathrm{Sp}, \otimes, \mathbb{S}) \rightsquigarrow$ (commutative) rings and modules

Prime Decomposition

Paradigm

Study at each prime, glue the results

Theorem

For $X \in \mathrm{Ab}$ there is a pullback square

$$\begin{array}{ccc} X & \longrightarrow & \prod X_p^\wedge \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & (\prod X_p^\wedge)_{\mathbb{Q}} \end{array}$$

- Controlled by the topology of $\mathrm{Spec}(\mathbb{Z})$
- More generally, for $R \in \mathrm{CRing}$, controlled by $\mathrm{Spec}(R)$

Balmer Spectrum

$$\mathcal{C} \longmapsto \mathrm{Spec}(\mathcal{C})$$

- $\mathcal{C}$ – symmetric monoidal stable category
- $\mathrm{Spec}(\mathcal{C})$ – topological space of “prime ideals”

Theorem

For $R \in \mathrm{CRing}$,

$$\underbrace{\mathrm{Spec}(\mathcal{D}(R))}_{\textit{(Balmer)}} \cong \underbrace{\mathrm{Spec}(R)}_{\textit{(Zariski)}}$$

- Can apply to $\mathcal{C} = \mathrm{Sp}$!

Chromatic Primes

Theorem (Hopkins-Devinatz-Smith)

The Balmer spectrum of $\mathrm{Sp}_{(p)}$ *is*

$$\mathrm{Sp}_{\mathbb{Q}} \to \underbrace{\mathrm{Sp}_{\mathrm{T}(1)} \to \mathrm{Sp}_{\mathrm{T}(2)} \to \cdots \to \mathrm{Sp}_{\mathrm{T}(n)} \to \cdots \to \mathrm{Sp}_{\mathbb{F}_p}}_{\mathrm{Sp}_p^\wedge}$$

- n is called the *height*
- Localizations $L_{\mathrm{T}(n)} : \mathrm{Sp} \to \mathrm{Sp}_{\mathrm{T}(n)}$

Examples

Example

Topological K-theory $\mathrm{KU}_p^\wedge \in \mathrm{Sp}_{\mathrm{T}(1)}$

Example

For any n, Lubin-Tate spectrum $\mathrm{E}_n \in \mathrm{Sp}_{\mathrm{T}(n)}$ (e.g. $\mathrm{E}_1 = \mathrm{KU}_p^\wedge$)

Algebraic K-Theory

Group-complete K-theory (baby case)

- Stable category $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$
- Space of objects $\mathcal{C}^{\simeq} \in \mathrm{CMon}(\mathcal{S})$ (w.r.t direct sum)
- Group complete to get $(\mathcal{C}^{\simeq})^{\mathrm{gpc}} \in \mathrm{CMon}^{\mathrm{gl}}(\mathcal{S}) = \mathrm{Sp}_{\geq 0}$

Algebraic K-theory

- Neglected stable structure – (co)fiber sequences
- Force $Y = X + Z$ for $X \to Y \to Z$ ($S_{\bullet}$-construction)

$$\mathrm{K}\colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$$

Example

$\mathrm{K}_0(\mathcal{C}) = (\pi_0 \mathcal{C}^{\simeq})^{\mathrm{gpc}} / \{Y = X + Z\}$

K-Theory of Rings

Definition

For $R \in \mathrm{Alg}(\mathrm{Sp})$ (e.g. $R \in \mathrm{Alg}(\mathrm{Ab})$), we let

$$\mathrm{K}(R) := \mathrm{K}(\mathrm{Mod}_R^{\mathrm{dbl}}(\mathrm{Sp}))$$

Example

$\mathrm{K}(\mathbb{C})_p^\wedge = \mathrm{ku}_p^\wedge$ (same for $\overline{\mathbb{Q}}$)

- $\mathbb{C}$ has height 0
- $\mathrm{K}(\mathbb{C})$ has height 1

Redshift Conjecture (Ausoni-Rognes)

Conjecture

Let R have height n, then $\mathrm{K}(R)$ has height $\leq n+1$, i.e.

- $L_{\mathrm{T}(m)}\mathrm{K}(R) = 0$ *for* $m > n+1$
- $L_{\mathrm{T}(n+1)}\mathrm{K}(R) \neq 0$

- Until recently, only in examples for $n \leq 1$
- First part by Land-Mathew-Meier-Tamme + Clausen-Mathew-Naumann-Noel
- Second part by Hahn-Wilson and Yuan (examples for all n)

Higher Semiadditivity in Vector Spaces

Vect_k is always

- Pointed (initial object = terminal object)
- Semiadditive (finite coproducts = finite products)

Definition

Let $G \in \mathrm{Grp}$ act on $V \in \mathrm{Vect}_k$, define $\mathrm{Nm}\colon V_G \to V^G$ by $\mathrm{Nm}([x]) := \sum_g gx$

- If $|G|$ is invertible in k then Nm is an isomorphism
- Otherwise, can be 0
- $V_G = \mathrm{colim}_{\mathrm{B}G} V$ and $V^G = \lim_{\mathrm{B}G} V$

Corollary

If $\mathrm{char}(k) = 0$ then colimits = limits over finite groupoids

Higher Semiadditivity

Definition

$A \in \mathcal{S}$ is m-*finite* if

- $\pi_0 A$ is finite
- $\pi_i(A, a)$ is finite for any $a \in A$
- $\pi_i(A, a) = 0$ for $i > m$

$A \in \mathcal{S}$ is π-*finite* if it is m-finite for some m

Example

- (-1)-finite = $\emptyset, *$
- 0-finite = finite set
- 1-finite = finite coproduct of $\mathrm{B}G$'s where G is finite

Higher Semiadditivity

Definition (Hopkins-Lurie)

$\mathcal{C}$ is m-*semiadditive* if for any m-finite $A \in \mathcal{S}$ and $X : A \to \mathcal{C}$

$$\operatorname*{colim}_A X \xrightarrow{\mathrm{Nm}} \lim_A X$$

is an isomorphism

Example

- (-1)-semiadditive = pointed
- 0-semiadditive = semiadditive

Higher Semiadditivity in Vector Spaces

Example

$\mathrm{Vect}_{\mathbb{Q}}$ is ∞-semiadditive

Example

$\mathrm{Vect}_{\mathbb{F}_p}$ is 0-semiadditive but *not* m-semiadditive for $m \geq 1$

Higher Semiadditivity in Chromatic Homotopy

Theorem (Carmeli-Schlank-Yanovski)

$\mathrm{Sp}_{\mathrm{T}(n)}$ *is* ∞*-semiadditive for any height* $n < \infty$

Known previously

- $\mathrm{Sp}_{\mathbb{Q}}$ is ∞-semiadditive (easy)
- $\mathrm{Sp}_{\mathrm{T}(n)}$ is 1-semiadditive (Kuhn)
- $\mathrm{Sp}_{\mathrm{K}(n)} \subset \mathrm{Sp}_{\mathrm{T}(n)}$ is ∞-semiadditive (Hopkins-Lurie)

Higher Commutative Monoids

- Commutative monoid $X \in \mathrm{CMon}(\mathcal{C})$
  - Summation of finite families of elements $\sum_A : X^A \to X$
  - Comm., assoc. and unital up to specified homotopies
- (Harpaz) ∞-commutative monoid $X \in \mathrm{CMon}_\infty(\mathcal{C})$
  - “Integration” of π-finite families of elements $\int_A : X^A \to X$
  - Comm., assoc. and unital up to specified homotopies

Example (Harpaz)

If $\mathcal{C}$ is ∞-semiadditive, every object $X \in \mathcal{C}$ is canonically an ∞-commutative monoid with integration

$$\lim_A X \xrightarrow{\mathrm{Nm}^{-1}} \operatorname{colim}_A X \xrightarrow{\nabla} X$$

Higher Commutative Monoids

Example (Cocartesian structure)

$\mathrm{Cat}_{\pi\text{-fin}}$ is ∞-semiadditive, every object $\mathcal{C} \in \mathrm{Cat}_{\pi\text{-fin}}$ is canonically an ∞-commutative monoid with integration

$$\operatorname*{colim}_A \colon \mathcal{C}^A \to \mathcal{C}$$

Corollary

If $\mathcal{C}$ is ∞-semiadditive, every object $X \in \mathcal{C}$ is an ∞-commutative monoid in $\mathcal{C}$, and $\mathcal{C}$ itself is an ∞-commutative monoid in $\mathrm{Cat}_{\pi\text{-fin}}$

Higher Commutative Monoids

Proposition

$\mathrm{CMon}(\mathcal{S})$ *is the universal presentable semiadditive category*

Theorem (Harpaz)

$\mathrm{CMon}_\infty(\mathcal{S})$ *is the universal presentable* ∞*-semiadditive category*

Definition

$\beth := \mathrm{CMon}_\infty(\mathrm{Sp})$, the universal presentable stable ∞-semiadditive category

Example

$\beth \to \mathrm{Sp}_{\mathrm{T}(n)}$, with fully faithful right adjoint

Semiadditive Height

- $X \in \mathrm{Sp}_{(p)}$ is of height
  - 0 if p-invertible
  - > 0 if p-complete
- Multiplication-by-p = $x \mapsto \sum_p x$
- Multiplication-by-$|A|$ = $x \mapsto \int_A x$

Definition (Carmeli-Schlank-Yanovski)

We say $X \in \mathcal{C}$ has *semiadditive height* $\mathrm{ht}(X)$

- $\leq n$ if $|\mathrm{B}^n C_p|$-invertible
- $> n$ if $|\mathrm{B}^n C_p|$-complete

Semiadditive Height

Example

Every object $X \in \mathrm{Sp}_{\mathrm{T}(n)}$ is of semiadditive height $\mathrm{ht}(X) = n$

Proposition

If $F\colon \mathcal{C} \to \mathcal{D}$ is ∞-semiadditive (preserves (co)limits over π-finite spaces), then $\mathrm{ht}(X) \leq n$ implies $\mathrm{ht}(FX) \leq n$

Semiadditive Redshift

Theorem (Carmeli-Schlank-Yanovski)

Let $\mathcal{C}$ be ∞-semiadditive, then TFAE

- $\mathrm{ht}(X) \leq n$ *for any* $X \in \mathcal{C}$
- $\mathrm{ht}(\mathcal{C}) \leq n+1$ *(as an object of* $\mathrm{Cat}_{\pi\text{-fin}}$*)*

Semiadditive K-Theory

Slogan

Carry ∞-commutative monoid structure along K-theory

Group-complete semiadditive K-theory (baby case)

- Stable category with π-finite colimits $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}_{\pi\text{-fin}}$
- Space of objects $\mathcal{C}^{\simeq} \in \mathrm{CMon}_{\infty}(\mathcal{S})$ (w.r.t colimits)
- Group complete to get $(\mathcal{C}^{\simeq})^{\mathrm{gpc}} \in \text{צ}$

Semiadditive K-theory

- Again, neglected (co)fiber sequences
- $S_{\bullet}$-construction preserves ∞-commutative monoids

$$\mathrm{K}^{[\infty]} \colon \mathrm{Cat}^{\mathrm{st}}_{\pi\text{-fin}} \to \text{צ}$$

Redshift?

Proposition

$\mathrm{K}^{[\infty]}\colon \mathrm{Cat}^{\mathrm{st}}_{\pi\text{-fin}} \to ץ$ *is an* ∞*-semiadditive functor (preserves (co)limits over* π*-finite spaces)*

Corollary

If $\mathrm{ht}(\mathcal{C}) \leq n$ *then* $\mathrm{ht}(\mathrm{K}^{[\infty]}(\mathcal{C})) \leq n$

- Evidently, this does not exhibit redshift

Redshift!

- Categorification!
- Let $R \in \mathrm{Alg}(\mathrm{Sp}_{\mathrm{T}(n)})$, then $\mathrm{Mod}_R^{\mathrm{dbl}}(\mathrm{Sp}_{\mathrm{T}(n)})$ is ∞-semiadditive

Definition

$$\mathrm{K}^{[\infty]}(R) := \mathrm{K}^{[\infty]}(\mathrm{Mod}_R^{\mathrm{dbl}})$$

Corollary (of semiadditive redshift)

$$\mathrm{ht}(\mathrm{Mod}_R^{\mathrm{dbl}}) \leq n+1$$

Theorem (B.M.-Schlank)

$$\mathrm{ht}(\mathrm{K}^{[\infty]}(R)) \leq n+1$$

Cyclotomic Extensions

- $\mathrm{ht}(\mathrm{Mod}_R^{\mathrm{dbl}}) = n+1$ controlled by colimits over $\mathrm{B}^n C_p$
- Leads to consider $R[\mathrm{B}^n C_p] := \mathrm{colim}_{\mathrm{B}^n C_p} R$

Theorem (Carmeli-Schlank-Yanovski)

Let $R \in \mathrm{Alg}(\mathrm{Sp}_{\mathrm{T}(n)})$, there is a splitting

$$R[\mathrm{B}^n C_p] = R \times R[\omega_p^{(n)}]$$

- For rational ring, $R[\omega_p^{(0)}]$ is the p-cyclotomic extension
- $R[\omega_p^{(n)}]$ behaves analogously

Roots of Unity and Redshift

Definition

We say that $R \in \mathrm{Alg}(\mathrm{Sp}_{\mathrm{T}(n)})$ *has (height n) p-th roots of unity* if $R[\omega_p^{(n)}] = \prod_{p-1} R$

Theorem (B.M.-Schlank)

If R has p-th roots of unity then $\mathrm{ht}(\mathrm{K}^{[\infty]}(R)) = n + 1$

Example (Carmeli-Schlank-Yanovski)

E_n has p-th roots unity, thus $\mathrm{ht}(\mathrm{K}^{[\infty]}(\mathrm{E}_n)) = n + 1$

Relationship to Ordinary K-Theory

- In general, there is a map $\mathrm{K}(R) \to \mathrm{K}^{[\infty]}(R)$
- Gives $L_{\mathrm{T}(n+1)}\,\mathrm{K}(R) \to L_{\mathrm{T}(n+1)}\,\mathrm{K}^{[\infty]}(R)$
  - When is $\mathrm{K}^{[\infty]}(R)$ in the full subcategory $\mathrm{Sp}_{\mathrm{T}(n+1)} \subset$ צ?
  - When is this $\mathrm{T}(n+1)$-equivalence?
- $n = 0$
  - Quillen-Lichtenbaum conjecture for $\mathbb{S}[p^{-1}]$
  - Clausen-Mathew-Naumann-Noel

Theorem (B.M.-Schlank)

Let $R \in \mathrm{Alg}(\mathrm{Sp}[p^{-1}])$ *then* $\mathrm{K}^{[\infty]}(R) = L_{\mathrm{T}(1)}\,\mathrm{K}(R)$

Example

$\mathrm{K}^{[\infty]}(\overline{\mathbb{Q}}) = \mathrm{KU}_p^{\wedge}$

Things I Didn't Talk About

- Multiplicative structure
- Atomic objects (and monoidal Yoneda embedding)
- m-semiadditive K-theory

Further Directions

- $L_{\mathrm{T}(n+1)}\,\mathrm{K}$ and $\mathrm{K}^{[\infty]}$ (Carmeli-Schlank-Yanovski)
- Semiadditive Grothendieck-Witt (Carmeli-Yuan)
- Blumberg-Gepner-Tabuada type universal property

Thank You!