Higher Semiadditive Algebraic K-Theory and Redshift

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Implicitly, whenever I say category I mean ∞-category, etc.

1 Background on Redshift

1.1 Algebraic K-Theory

Let me begin by briefly reminding what is algebraic K-theory, in a way amenable to generalization later. Given a stable category $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}$, the space of objects is a commutative monoid with respect to direct sums $\mathcal{C}^{\simeq} \in \mathrm{CMon}(\mathcal{S})$. Now, a commutative monoid can be group-completed by applying the left adjoint to the inclusion

$$(-)^{\mathrm{gpc}} \colon \mathrm{CMon}(\mathcal{S}) \rightleftarrows \mathrm{CMon}^{\mathrm{gl}}(\mathcal{S}) = \mathrm{Sp}_{\geq 0},$$

just like the passage from $\mathbb{N} \mapsto \mathbb{Z}$. This gives us the direct sum K-theory

$$(\mathcal{C}^{\simeq})^{\mathrm{gpc}} \in \mathrm{Sp}_{\geq 0}.$$

However, we have only used the fact that $\mathcal{C}$ has direct sum, i.e., semiadditive, and not stability. Namely, we neglected (co)fiber sequences. Algebraic K-theory is obtained by doing the above while also forcing $Y = X + Z$ for any (co)fiber sequence $X \to Y \to Z$, using, for example, the $S_\bullet$-construction. This assembles into a functor

$$\mathrm{K} \colon \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0}.$$

Definition 1. For a ring spectrum $R \in \mathrm{Alg}(\mathrm{Sp})$ (e.g., any ordinary ring) we let

$$\mathrm{K}(R) := \mathrm{K}(\mathrm{Mod}_R^{\mathrm{dbl}}(\mathrm{Sp})).$$

Example 2. $\mathrm{K}(\mathbb{C})_p^\wedge = \mathrm{ku}_p^\wedge$.

1.2 Chromatic Homotopy

A very useful paradigm in ordinary algebra is studying questions one prime at a time and then gluing the results. For simplicity, let us work p-locally (i.e., keep rational and characteristic p information, and ignore all characteristic $q \neq p$ information), then this decomposition is controlled by the fairly simple topological space

$$\mathrm{Spec}(\mathbb{Z}_{(p)}) = \{(0) \to (p)\}.$$

The chromatic picture shows that over the sphere spectrum, there are new characteristics

$$\mathrm{Spec}(\mathbb{S}_{(p)}) = \{(0) \to (p,1) \to \cdots \to (p,n) \to \cdots \to (p,\infty)\}.$$

More specifically, there are localizations $L_{\mathrm{T}(n)}\colon \mathrm{Sp} \to \mathrm{Sp}_{\mathrm{T}(n)}$ for every n (the case $n = 0$ reproduces rationalization), where the number n is called the *height*. Generally speaking, the information gets more complicated as the height increases.

Example 3. Topological K-theory is of height 1, $\mathrm{KU}_p^\wedge \in \mathrm{Sp}_{\mathrm{T}(1)}$.

1.3 Redshift

As we mentioned above, $\mathrm{K}(\mathbb{C})_p^\wedge = \mathrm{ku}_p^\wedge$, which shows that algebraic K-theory takes something of height 0 (as $\mathbb{C}$ is rational) to something of height 1. This, along with more evidence when the input has height 1, has led Ausoni–Rognes to the far-reaching redshift conjecture. Their conjecture takes a specific strong form, out of which emerged a wider philosophy which can be loosely described as follows:

Conjecture 4. *Algebraic K-theory increases chromatic height by* 1.

Let me give one manifestation of this philosophy, which is not the original one, and was recently proven by a combination of breakthroughs. By a theorem of Hahn, if $R \in \mathrm{CAlg}(\mathrm{Sp})$ has $L_{\mathrm{T}(k)}R = 0$, then $L_{\mathrm{T}(k+1)}R = 0$ as well, thus R is supported on $0, \dots, n$ for some n.

Theorem 5. *Let $R \in \mathrm{CAlg}(\mathrm{Sp})$ be supported on $0, \dots, n$, then $\mathrm{K}(R)$ is supported on $0, \dots, n+1$.*

Clausen–Mathew–Naumann–Noel and Land–Mathew–Meier–Tamme showed that the support is at most $n + 1$. The other inequality was proven for examples at every height n by Hahn–Wilson and Yuan, and, building on that, for any R by Burklund–Schlank–Yuan.

2 Background on Higher Semiadditivity

2.1 m-Semiadditivity

Let us begin with ordinary algebra. The (1-)category Vect_k is

- pointed, i.e., initial object = terminal object,
- semiadditive, i.e., finite coproducts = finite products.

Because it is semiadditive, we can sum, which allows us to give the following definition.

Definition 6. Let G be a finite group acting on $V \in \mathrm{Vect}_k$, we let

$$\mathrm{Nm}\colon \underbrace{V_G}_{=\mathrm{colim}_{BG} V} \to \underbrace{V^G}_{=\lim_{BG} V}, \quad \mathrm{Nm}([x]) := \sum_{g\in G} gx.$$

Observe that if $|G|$ is invertible in k, then $\frac{1}{|G|}$ is an inverse to Nm, thus

Proposition 7. *If* $\mathrm{char}(k) = 0$*, then colimits = limits over finite groupoids. That is, let A be a finite groupoid and $X : A \to \mathrm{Vect}_k$ be a diagram, then*

$$\mathrm{Nm}\colon \operatorname*{colim}_A X \xrightarrow{\sim} \lim_A X.$$

On the other hand, for example for $k = \mathbb{F}_p$ and $G = C_p$ acting trivially, we have $\mathrm{Nm} = 0$, so this phenomenon does *not* happen at characteristic p for p-groups (but it does for prime-to-p groups). Surprisingly, this result, and a vast generalization of it, *does* hold in the intermediate characteristics $\mathrm{Sp}_{\mathrm{T}(n)}$ for $n \geq 1$, eventhough they are p-complete, which we now move on to.

Definition 8. We say that a space A is an *m-finite* p-space if

1. $\pi_0 A$ is finite,
2. $\pi_i(A, a)$ is a finite p-group for every $a \in A$,
3. A is m-truncated, i.e., $\pi_i(A, a) = 0$ for $i > m$.

From now on I am going to implicitly assume that all m-finite spaces are p-spaces.

Example 9. We have

- (-1)-finite $= \emptyset, *$.
- 0-finite = finite set.
- 1-finite = finite coproduct of BG's where G is finite.

Definition 10. $\mathcal{C}$ is (p-typically) m-semiadditive if for any m-finite space A and $X : A \to \mathcal{C}$ the norm map is an isomorphism

$$\mathrm{Nm}\colon \operatorname*{colim}_A X \xrightarrow{\sim} \lim_A X.$$

In these terms, what we have seen before is that $\mathrm{Vect}_{\mathbb{F}_p}$ is 0-semiadditive but not 1-semiadditive, $\mathrm{Vect}_{\mathbb{Q}}$ is 1-semiadditive (and as a 1-category, automatically ∞-semiadditive). Following a line of results by Greenlees–Hovey–Sadofsky, Kuhn and Hopkins–Lurie, we have:

Theorem 11 (Carmeli–Schlank–Yanovski)**.** $\mathrm{Sp}_{\mathrm{T}(n)}$ *is* ∞*-semiadditive.*

2.2 Higher Commutative Monoids

Higher semiadditivity gives a lot of extra structure and properties on the category, and I would like to emphasize one aspect, due to Harpaz. Let $\mathcal{C}$ be a (0-)semiadditive category, then, as mentioned before, every object $X \in \mathcal{C}$ is canonically a commutative monoid. Namely, there are summation maps $\sum_A \colon X^A \to X$ for any finite set A, which are coherently commutative, associative and unital. Harpaz defined the notion of an m-commutative monoid, where one has similar "integration" maps $\int_A \colon X^A \to X$ for an m-finite p-space A, which are coherently commutative, associative and unital.

Example 12. Let $\mathcal{C}$ be m-semiadditive, then every $X \in \mathcal{C}$ is canonically an m-commutative monoid with

$$\int_A \colon X^A = \lim_A X \xrightarrow{\mathrm{Nm}^{-1}} \operatorname*{colim}_A X \xrightarrow{\nabla} X.$$

Theorem 13 (Harpaz)**.** $\mathrm{CMon}_m(\mathcal{S})$ *is the universal presentable* m*-semiadditive category.*

Definition 14. $\mathfrak{Z}^{[m]} := \mathrm{CMon}_m(\mathrm{Sp})$ is the universal presentable m-semiadditive *stable* category.

Example 15. There is a canonical (smashing localization) functor $L^{\mathfrak{Z}^{[m]}}_{\mathrm{T}(n)} \colon \mathfrak{Z}^{[m]} \to \mathrm{Sp}_{\mathrm{T}(n)}$.

I would like to give one more example of an m-semiadditive category, which will play a significant role later.

Example 16. The category $\mathrm{Cat}_{m\text{-fin}}$ of categories that have m-finite (p-space) colimits and functors preserving them is m-semiadditive. As such, every $\mathcal{C} \in \mathrm{Cat}_{m\text{-fin}}$ is itself canonically an m-commutative monoid, with integration maps given by

$$\operatorname*{colim}_A \colon \mathcal{C}^A \to \mathcal{C}.$$

2.3 Semiadditive Height

Carmeli–Schlank–Yanovski observed that height can be measured using the commutative monoid structure on $X \in \mathcal{C}$. In the interest of time, I won't give the definition, but I will mention that, just like for spectra, an object can be supported at different heights. Using the integration maps for $A = \mathrm{B}^n C_p$, they define when the *semiadditive height* $\mathrm{ht}(X)$ is $\leq n$ or $> n$.

Example 17. Every object $X \in \mathrm{Sp}_{\mathrm{T}(n)}$ has semiadditive height n.

Proposition 18. *If* $F \colon \mathcal{C} \to \mathcal{D}$ *is* m*-semiadditive (preserves (co)limits over* m*-finite* p*-spaces), then* $\mathrm{ht}(X) \leq n$ *implies* $\mathrm{ht}(F(X)) \leq n$*.*

If $\mathcal{C}$ is m-semiadditive, then we can measure the height of every object $X \in \mathcal{C}$. On the other hand, since $\mathcal{C} \in \mathrm{Cat}_{m\text{-fin}}$ is itself an object of an m-semiadditive category, we can measure *its* height.

Theorem 19 (Semiadditive Redshift, Carmeli–Schlank–Yanovski)**.** *Let* $\mathcal{C}$ *be* m*-semiadditive, then TFAE*

- $\mathrm{ht}(X) \leq n$ *for every* $X \in \mathcal{C}$*,*
- $\mathrm{ht}(\mathcal{C}) \leq n+1$*, as an object of* $\mathrm{Cat}_{m\text{-fin}}$*.*

3 Higher Semiadditive K-Theory

3.1 Definition

Recall that to define algebraic K-theory, we observed that since the category is semiadditive the space of objects is a commutative monoid, which we then group-completed to get $(\mathcal{C}^{\simeq})^{\mathrm{gpc}} \in \mathrm{Sp}_{\geq 0}$. To actually get algebraic K-theory we also need to split (co)fiber sequences. Now, if $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}_{m\text{-fin}}$, then as above the space of objects is an m-commutative monoid. We can thus take the group-completion while preserving this structure, namely apply the canonical functor

$$(-)^{\mathrm{gpc}} \colon \mathrm{CMon}_m(\mathcal{S}) \to \mathrm{CMon}_m(\mathrm{Sp}) = \mathfrak{Z}^{[m]},$$

giving an m-semiadditive version of direct sum K-theory. Again, we also want to split (co)fiber sequences, which we implement using the $S_\bullet$-construction, resulting in a functor

$$\mathrm{K}^{[m]} \colon \mathrm{Cat}^{\mathrm{st}}_{m\text{-fin}} \to \mathfrak{Z}^{[m]}.$$

The case $m = 0$ reproduces (p-localized) ordinary algebraic K-theory. From now on, I always assume $m \geq 1$.

Definition 20. For a ring spectrum $R \in \mathrm{Alg}(\mathrm{Sp}_{\mathrm{T}(n)})$, we show that $\mathrm{Mod}^{\mathrm{dbl}}_R(\mathrm{Sp}_{\mathrm{T}(n)})$ is ∞-semiadditive, which allows us to define

$$\mathrm{K}^{[m]}(R) := \mathrm{K}^{[m]}(\mathrm{Mod}^{\mathrm{dbl}}_R(\mathrm{Sp}_{\mathrm{T}(n)})).$$

$\mathrm{K}^{[m]}$ is a functor between m-semiadditive categories, and we show the following:

Proposition 21. $\mathrm{K}^{[m]} \colon \mathrm{Cat}^{\mathrm{st}}_{m\text{-fin}} \to \mathfrak{Z}^{[m]}$ *is m-semiadditive.*

In fact, more is true – we show that $\mathrm{K}^{[m]}$ is in a sense obtained from K by forcing it to be m-semiadditive.

3.2 Semiadditive Redshift

Recall that m-semiadditive functors can only decrease height, thus we conclude that

Proposition 22. *If* $\mathrm{ht}(\mathcal{C}) \leq n$ *then* $\mathrm{ht}(\mathrm{K}^{[m]}(\mathcal{C})) \leq n$.

This does not look like redshift, instead, redshift happens at the stage of *categorification.*

Theorem 23. *Let* $R \in \mathrm{Alg}(\mathrm{Sp}_{\mathrm{T}(n)})$, *then* $\mathrm{ht}(\mathrm{K}^{[m]}(R)) \leq n + 1$.

Proof sketch. Recall that all objects of $\mathrm{Sp}_{\mathrm{T}(n)}$ have semiadditive height n, from which the same follows for $\mathrm{Mod}^{\mathrm{dbl}}_R$. By the semiadditive redshift theorem, we get that $\mathrm{ht}(\mathrm{Mod}^{\mathrm{dbl}}_R) \leq n + 1$ as an object of $\mathrm{Cat}^{\mathrm{st}}_{m\text{-fin}}$, so the result follows from the previous proposition. ☐

Using the higher commutative monoid structure, Carmeli–Schlank–Yanovski defined height n analogues of *cyclotomic extensions* $R[\omega_p^{(n)}]$ (which for $n = 0$ reproduce ordinary cyclotomic extensions). Using these, one can say that R has *height n p-th roots of unity*, if $R[\omega_p^{(n)}] = \prod_{(\mathbb{Z}/p)^\times} R$, which is satisfied, for example, for the Lubin–Tate spectrum E_n.

Theorem 24. *If* $R \in \mathrm{Alg}(\mathrm{Sp}_{\mathrm{T}(n)})$ *has height n p-th roots of unity, then* $\mathrm{ht}(\mathrm{K}^{[m]}(R)) = n + 1$.

3.3 Relationship to Chromatically Localized K-Theory

We have seen that higher semiadditive K-theory satisfies a form of redshift for semiadditive height. First, note that semiadditive height $n + 1$ can only be measured when $m \geq n + 1$. Second, it would be interesting to connect it chromatically localized K-theory, which would in particular allow us to measure height without assuming $m \geq n + 1$, addressing the first issue.

Recall that I have mentioned that $\mathrm{K}^{[m]}$ can be obtained from algebraic K-theory by forcing it to be m-semiadditive. Also recall that since $\mathrm{Sp}_{\mathrm{T}(n+1)}$ is ∞-semiadditive, there is a (smashing) localization $L^{\mathfrak{Y}^{[m]}}_{\mathrm{T}(n+1)} \colon \mathfrak{Y}^{[m]} \to \mathrm{Sp}_{\mathrm{T}(n+1)}$. Using these ideas, for $\mathcal{C} \in \mathrm{Cat}^{\mathrm{st}}_{m\text{-fin}}$, we construct a comparison map

$$L_{\mathrm{T}(n+1)}\,\mathrm{K}(\mathcal{C}) \to L^{\mathfrak{Y}^{[m]}}_{\mathrm{T}(n+1)}\,\mathrm{K}^{[m]}(\mathcal{C})$$

which is an isomorphism if and only if for every m-finite p-space A, the assembly map

$$L_{\mathrm{T}(n+1)}\,\mathrm{K}(\mathcal{C}^A) \xrightarrow{\sim} L_{\mathrm{T}(n+1)}\,\mathrm{K}(\mathcal{C})^A$$

is an isomorphism. As we see, this is very closely related to descent for chromatically localized K-theory.

Theorem 25 (Clausen–Mathew–Naumann–Noel)**.** *The functor*

$$\mathrm{Cat}_{L_n^f} \xrightarrow{\mathrm{K}} \mathrm{Sp} \xrightarrow{L_{\mathrm{T}(n+1)}} \mathrm{Sp}_{\mathrm{T}(n+1)}$$

commutes with limits indexed by 1*-finite p-spaces.*

Corollary 26. *Let $\mathcal{C} \in \mathrm{Cat}_{L_n^f,1\text{-fin}}$ (e.g.,* $\mathrm{Mod}^{\mathrm{dbl}}_R$ *for $R \in \mathrm{Alg}(\mathrm{Sp}_{\mathrm{T}(n)})$), then*

$$L^{\mathfrak{Y}^{[1]}}_{\mathrm{T}(n+1)}\,\mathrm{K}^{[1]}(\mathcal{C}) = L_{\mathrm{T}(n+1)}\,\mathrm{K}(\mathcal{C}).$$

In upcoming work with Carmeli, Schlank and Yanovski, we generalize this result to arbitrary m:

Theorem 27. *The functor*

$$\mathrm{Cat}_{L_n^f} \xrightarrow{\mathrm{K}} \mathrm{Sp} \xrightarrow{L_{\mathrm{T}(n+1)}} \mathrm{Sp}_{\mathrm{T}(n+1)}$$

commutes with limits indexed by m-finite p-spaces.

Corollary 28. *Let $\mathcal{C} \in \mathrm{Cat}_{L_n^f,m\text{-fin}}$, then*

$$L^{\mathfrak{Y}^{[m]}}_{\mathrm{T}(n+1)}\,\mathrm{K}^{[m]}(\mathcal{C}) = L_{\mathrm{T}(n+1)}\,\mathrm{K}(\mathcal{C}).$$

Moreover, this result allows us to transport higher semiadditive constructions through chromatically localized K-theory, for example, cyclotomic extensions:

Corollary 29. *Let $R \in \mathrm{Alg}(\mathrm{Sp}_{\mathrm{T}(n)})$, then*

$$L_{\mathrm{T}(n+1)}\,\mathrm{K}(R[\omega_p^{(n)}]) = L_{\mathrm{T}(n+1)}\,\mathrm{K}(R)[\omega_p^{(n+1)}].$$

These results show that higher semiadditive K-theory, when pushed to $\mathrm{Sp}_{\mathrm{T}(n+1)}$, agrees with chromatically localized K-theory. As we have seen before, in many cases $\mathrm{K}^{[m]}(\mathcal{C}) \in \mathfrak{Y}^{[m]}$ is of pure semiadditive height $n+1$. One may wonder if it is in fact in $\mathrm{Sp}_{\mathrm{T}(n+1)} \subset \mathfrak{Y}^{[m]}$. We show that this question is closely related to the Quillen–Lichtenbaum conjecture for R, in the guise of having a finite spectrum such that $\mathrm{K}(R) \otimes X$ is bounded above. Using the Quillen–Lichtenbaum property of $\mathbb{S}[p^{-1}]$, and the descent result above, we settle the case of height 0 for any $m \geq 1$:

Theorem 30. *Let $R \in \mathrm{Alg}(\mathrm{Sp}[p^{-1}])$, then*

$$\mathrm{K}^{[m]}(R) = L_{\mathrm{T}(1)}\,\mathrm{K}(R).$$

For example, $\mathrm{K}^{[m]}(\mathbb{C}) = \mathrm{KU}^{\wedge}_p$*.*

Finally, using Hahn–Wilson's Quillen–Lichtenbaum result for $\mathrm{BP}\langle n\rangle$, we also answer the question for the completed Johnson–Wilson spectrum $\widehat{\mathrm{E}(n)} \in \mathrm{Sp}_{\mathrm{T}(n)}$:

Theorem 31. *We have*

$$\mathrm{K}^{[m]}(\widehat{\mathrm{E}(n)}) = L_{\mathrm{T}(n+1)}\,\mathrm{K}(\widehat{\mathrm{E}(n)}).$$

(The case $m \geq 2$ depends on the upcoming work with Carmeli, Schlank and Yanovski.)

3.4 Further Directions

- We conjecture that the last result holds for any $R \in \mathrm{Alg}(\mathrm{Sp}_{\mathrm{T}(n)})$ and $m \geq 1$.
- Develop a Blumberg–Gepner–Tabuada type universal property for $\mathrm{K}^{[m]}$.
- Is splitting (co)fiber sequences needed.
- Semiadditive Grothendieck–Witt theory, as initiated by Carmeli–Yuan.