Descent and Redshift in Algebraic K-Theory

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1 Zeroth K-group

Let R be a ring, and consider the collection of isomorphism classes of finitely generated projective R-modules. The direct sum makes into a commutative monoid.

Definition 1.1. We define the 0-th algebraic K group of R as the group-completion

$$\mathrm{K}_0(R) := (\mathrm{Proj}_R/\sim, \oplus)^{\mathrm{gpc}} \qquad \in \qquad \mathrm{Ab}.$$

Here the operation of group-completion is the left adjoint of the inclusion

$$\mathrm{Ab} \hookrightarrow \mathrm{CMon}.$$

Example 1.2. In a PID (e.g. fields and $\mathbb{Z}$) projective implies free, thus

$$\mathrm{K}_0(R) \simeq \mathbb{N}^{\mathrm{gpc}} \simeq \mathbb{Z}.$$

Example 1.3. More generally, for a Dedekind domain we have

$$\mathrm{K}_0(R) \simeq \mathbb{Z} \oplus \mathrm{Pic}(R).$$

2 Homotopy theory

In our construction of $\mathrm{K}_0(R)$, we took the isomorphism classes of modules, neglecting the interesting information encoded in their automorphisms, i.e. in the groupoid $\mathrm{Proj}_R^{\simeq}$. This is analgoues to excluding stacky phenomena. As the notation suggests, there is an infinite series of groups $\mathrm{K}_n(R)$, originally constructed by Quillen. Importantly, these K groups are in fact the homotopy groups of a much richer object, called a spectrum, which behaves much better as a whole. It is somewhat difficult to give a formal definition of spectra, but I will give two sets of intuitions.

Another source of groupoids is homotopy types. Given a topological space X, we can approximate it to 0-th order by its set of connected componenets. As a better approximation, we can consider the fundamental groupoid $\Pi_1(X)$, whose objects are the points $x \in X$, and morphisms are paths $[0,1] \to X$ up to homotopy between them (note that the automorphisms of $x \in X$ are precisely the fundamental group $\pi_1(X, x)$). We may continue, and consider the 2-groupoid $\Pi_2(X)$, whose objects are $x \in X$, morphisms are $[0,1] \to X$, and morphisms between morphisms are homotopies $[0,1]^2 \to X$ up to homotopy between them. This process continues, and we can define the ∞-groupoid $\Pi_\infty(X)$, capturing the homotopy type of X.

Going back to our case, note that on the groupoid $\mathrm{Proj}_R^{\simeq}$, we have the direct sum operation, in the sense that we can also form the direct sum of maps. In a similar way, we may consider commutative monoid, and abelian groups, in ∞-groupoids. With this in mind we let

$$\mathrm{Sp}_{\geq 0} := \mathrm{Ab}(\mathrm{Grpd}_\infty).$$

This is very closely related to derived categories:

- The derived category $\mathcal{D}(\mathbb{Z})$ has a subcategory $\mathcal{D}(\mathbb{Z})_{\geq 0}$ of non-negative objects. The shift operation on the latter is only invertible from one side, and inverting it recovers $\mathcal{D}(\mathbb{Z})$. In exactly the same way, we can form Sp from $\mathrm{Sp}_{\geq 0}$.
- Given an abelian group A, we have corresponding object in $\mathcal{D}(\mathbb{Z})$ by placing it in degree 0, and similary for Sp.
- In the other direction, given $A \in \mathcal{D}(\mathbb{Z})$ we can form the (co)homology groups $H_*(A)$, and similarly given $A \in \mathrm{Sp}$ we can form its homotopy groups $\pi_*(A)$.
- $\mathcal{D}(\mathbb{Z})$ has a derived tensored product. Similarly, Sp has a tensor product with unit $\mathbb{S}$. This allows us to define (commutative) ring spectra, modules over them, etc.
- For an (ordinary) ring R, we get a corresponding ring spectrum, and we have $\mathrm{Mod}_R(\mathrm{Sp}) \simeq \mathcal{D}(R)$. From this perspective spectra can be thought of as providing a base $\mathbb{S}$ sitting even before $\mathbb{Z}$.

3 Algebraic K-theory

Consider the groupoid $\mathrm{Proj}_R^{\simeq}$ as an ∞-groupoid, together with the direct sum operation. Applying the left adjoint of the inclusion

$$\mathrm{Sp}_{\geq 0} := \mathrm{Ab}(\mathrm{Grpd}_\infty) \hookrightarrow \mathrm{CMon}(\mathrm{Grpd}_\infty)$$

we arrive at the full definition.

Definition 3.1. We define the algebraic K-theory spectrum of R to be

$$\mathrm{K}(R) := (\mathrm{Proj}_R^{\simeq}, \oplus)^{\mathrm{gpc}} \qquad \in \qquad \mathrm{Sp}_{\geq 0},$$

and we let $\mathrm{K}_n(R) := \pi_n(\mathrm{K}(R))$.

A point that we will return to later is that this definition extends as-is to ring spectra. However, one may wonder why should we care about ∞-groupoids, when $\mathrm{Proj}_R^{\simeq}$ is just a groupoid. The reason is that, just like a derived functor, the group completion may spread over all degrees. In general, K groups are notoriously difficult to calculate.

Example 3.2 (Bass–Milnor–Serre)**.** For a number field L

$$\mathrm{K}_1(\mathscr{O}_L) \simeq \mathscr{O}_L^\times.$$

Example 3.3. We have

$$\mathrm{K}_1(\mathbb{Z}) = \mathbb{Z}^\times = \mathbb{Z}/2, \qquad \mathrm{K}_2(\mathbb{Z}) = \mathbb{Z}/2, \qquad \mathrm{K}_3(\mathbb{Z}) = \mathbb{Z}/48, \qquad \mathrm{K}_4(\mathbb{Z}) = 0.$$

The Vandiver conjecture is equivalent to $\mathrm{K}_{4n}(\mathbb{Z}) = 0$, and the second case $\mathrm{K}_8(\mathbb{Z}) = 0$ was only proven in 2018.

This exemplifies the intricate arithmetic infromation encoded by K groups, for instance, they and are closely related to special values of zeta functions. As another manifestation of this principle, we have the now-proven Quillen–Lichtenbaum conjecture, a consequence of Voevodsky–Rost's proof of the Bloch–Kato conjecture.

Theorem 3.4. *Let R be a suitable ring (regular Noetherian finitely generated) in which p is invertible. Then, there is a spectral sequence with*

$$E_2^{**} = H^*_{\mathrm{et}}(R; \mathbb{Z}/p(\frac{*}{2}))$$

which converges to $\pi_*(\mathrm{K}(R)/p)$ for $* \gg 0$.

Remark 3.5. Note that it is important to take mod p before taking π_* (similar to taking derived cokernel).

Loosely speaking, there is some relation between K-theory and etale cohomology. This statement, however, is somewhat odd, with the connection holding only for $* \gg 0$, leading us to our next topic.

4 Descent

As we have seen, algebraic K-theory is a fairly intricate invariant, and is difficult to compute. One way to approach such calculations is via descent.

Theorem 4.1 (Thomason)**.** *K-theory satisfies Zariski descent. That is, for affine schemes $X = U \cup V$, we have a pullback square in spectra*

$$\begin{array}{ccc} \mathrm{K}(X) & \longrightarrow & \mathrm{K}(U) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{K}(V) & \longrightarrow & \mathrm{K}(U \cap V) \end{array}$$

Remark 4.2. Note that the pullback must be taken in spectra (like a derived pullback), as with taking mod p above. It is *not* true that each K_n satisfies descent. In more down to earth terms though, this gives rise to a long exact sequence on K groups.

In fact Thomason proved more than that: K-theory satisfies Nisnevich descent. The Nisnevich topology is intermediate between Zariski and etale. A map $\coprod U_i \to X$ is a Nisnevich cover if it is an etale cover and for each point $x \in X$ there exists some i and $u \in U_i$ s.t. $k(x) \xrightarrow{\sim} k(u)$ is an isomorphism. Notably, K-theory does *not* satisfy etale descent, precisely because it fails to satisfy Galois descent. That is, there examples of a G-Galois extension $L \to L'$ such that

$$\mathrm{K}(L) \longrightarrow \mathrm{K}(L')^{hG}$$

is not an isomorphism. This ties back to the peculiarities of the Quillen–Lichtenbaum conjecture, which can be remedied using chromatic homotopy theory.

5 Chromatic localizations

A very useful paradigm in ordinary algebra is studying questions one prime at a time and then gluing the results. For simplicity, let us work p-locally, then this decomposition is controlled by the fairly simple topological space

$$\mathrm{Spec}(\mathbb{Z}_{(p)}) = \{(0) \to (p)\}.$$

For exampe, a p-local abelian group A can be recovered from its rationalization and p-completion glued along the rationalization of the p-completion

$$A = A_{\mathbb{Q}} \underset{(A_p)_{\mathbb{Q}}}{\times} A_p.$$

Surprisingly, this picture refines in spectra – this is the subject of chromatic homotopy theory. The p-completion further decomposes into infinitely many "new characteristics"

$$\mathrm{Spec}(\mathbb{S}_{(p)}) = \{(0) \to (p,1) \to \cdots \to (p,n) \to \cdots \to (p,\infty)\}.$$

In particular, there is a localization $L_{\mathrm{T}(n)} \colon \mathrm{Sp} \to \mathrm{Sp}_{\mathrm{T}(n)}$ for every *height* $n \geq 0$, where the case of height is 0 is rationalization, while the higher heights are refinements of the p-completion (we remark that there is also some information not captured by any of the finite heights, but for our purposes it will negligible).

I will not give too many details, but let me comment that this comes from a very close connection between spectra and (1-dimensional) formal groups, which over $\overline{\mathbb{F}}_p$ are classified by a number called the height, due to deep insights of Quillen, Morava, Ravenel and Devinatz–Hopkins–Smith, among many others. We have the following other manifestation of this connection.

Example 5.1. The Lubin–Tate deformation theory of formal groups can be carried out in spectra: given formal group $\mathbf{G}$ of height n over a field L, there is a commutative ring spectrum $\mathrm{E}(L, \mathbf{G})$, whose π_0 is the ordinary Lubin–Tate ring, which is $\mathrm{T}(n)$-local $\mathrm{E}(L, \mathbf{G}) = L_{\mathrm{T}(n)}\mathrm{E}(L, \mathbf{G})$.

This is a new phenomenon, not available in ordinary or derived algebra, but only over spectra. Given an abelian group A considered as a spectrum $A \in \mathrm{Sp}$, we have $L_{\mathrm{T}(n)}A = 0$ for all $n \geq 1$. Note that this in particular shows that the chromatic localizations are insensitive to some form of p-torsion information – nevertheless, as we shall soon see, this difference will not be important for our puroses.

Going back to algebraic K-theory, we have the following fundamental theorem.

Theorem 5.2 (Mithcell). *For any ring R and $n \geq 2$ we have $L_{\mathrm{T}(n)}\mathrm{K}(R) = 0$.*

Furthermore, Thomason showed that the T(1)-local part is precisely the etale part, relevant for Quillen–Lichtenbaum.

Theorem 5.3 (Thomason). *The functor $L_{\mathrm{T}(1)}\mathrm{K}(-)$ satisfies Galois, hence etale, descent. In fact, it is the etale sheafification for p-invertible rings, for which we have a spectral sequences*

$$E_2^{**} = H^*_{\mathrm{et}}(R; \mathbb{Z}_p(\frac{*}{2})) \quad \Longrightarrow \quad \pi_*(L_{\mathrm{T}(1)}\mathrm{K}(R)).$$

Thus the Quillen–Lichtenbaum conjecture can be reformulated as saying that

$$\pi_*(\mathrm{K}(R)) \longrightarrow \pi_*(L_{\mathrm{T}(1)}\mathrm{K}(R))$$

is an isomorphism for $* \gg 0$.

6 Redshift

We have seen various phenomena at case of ordinary rings, namely rings of height 0 (as $L_{\mathrm{T}(n)}R = 0$ for $n \geq 1$). Base on this case, as well as some computational evidence for ring spectra of height 1, in the early 2000's Ausoni–Rognes proposed a cluster of conjectures, now known as the redshift conjecture. These roughly say that K-theory increases chromatic height by 1, and that one $\mathrm{T}(n+1)$-localize K-theory should have better descent properties. These have seen tremendous breakthroughs in recent years by works of various groups, including Land–Mathew–Meier–Tamme, Clausen–Mathew–Naumann–Noel and Burklund–Schlank–Yuan.

Theorem 6.1. *K-theory increases chromatic height by one: let R be a $\mathrm{T}(n)$-local commutative ring spectrum then*

$$L_{\mathrm{T}(n+1)}\mathrm{K}(R) \neq 0, \qquad L_{\mathrm{T}(m)}\mathrm{K}(R) = 0 \quad \forall m \geq n+2.$$

Theorem 6.2. $\mathrm{T}(n+1)$*-localized K-theory satisfies Galois descent for finite p-groups. That is, let $R \to S$ be a Galois extension of* $\mathrm{T}(n)$*-local rings with finite p-group Galois group G, then*

$$L_{\mathrm{T}(n+1)}\mathrm{K}(R) \xrightarrow{\sim} L_{\mathrm{T}(n+1)}\mathrm{K}(S)^{hG}.$$

7 Cyclotomic redshift and the telescope conjecture

Finally, I'd like to discuss joint work with Carmeli, Schlank and Yanovski. Our main theorem is an extension of this last result in a somewhat abstract direction, replacing the Galois group by a Galois n-groupoid. Instead of describing it, I will formulate some consequences that are somewhat more tangible, concerning analogues of cyclotomic extension.

Given a commutative ring R, we may consider the cyclotomic extension $R[\omega_{p^k}]$. We note that this construction works well away from p, i.e. when p is invertible in R, for instance it is a $(\mathbb{Z}/p^k)^\times$-Galois extension, while for $\mathbb{F}_p$-algebras it is totally ramified, and behaves completely differently. Surprisingly, while $\mathrm{T}(n)$-local spectra are in a sense at the prime p, they still have a working theory of cyclotomic extensions, due to Carmeli–Schlank–Yanovski. That is, for a $\mathrm{T}(n)$-local commutative ring spectrum R, there is a $(\mathbb{Z}/p^k)^\times$-Galois extension $R[\omega_{p^r}^{(n)}]$, called the height n cyclotomic extension. Furthermore, these support a theory analogues to discrete fourier transform, and Kummer theory. One of our main results is the following.

Theorem 7.1 (B.M.–Carmeli–Schlank–Yanovski)**.** $\mathrm{T}(n+1)$*-localized K-theory sends height n cyclotomic extension to height $n+1$ cyclotomic extensions, i.e.*

$$L_{\mathrm{T}(n+1)}\mathrm{K}(R[\omega_{p^r}^{(n)}]) \simeq L_{\mathrm{T}(n+1)}\mathrm{K}(R)[\omega_{p^r}^{(n+1)}]$$

together with the $(\mathbb{Z}/p^k)^\times$-action.

Note that $(\mathbb{Z}/p^k)^\times$ is not a p-group (e.g. $|(\mathbb{Z}/p)^\times| = p-1$), and hence doesn't fit in the previous theorem, and in particular gives the first prime-to-p example of Galois descent for $\mathrm{T}(n+1)$-localized K-theory.

Finally, we discuss the case of the infinite cyclotomic extension $R[\omega_{p^\infty}^{(n)}]$, which is a profinite Galois extension for $\mathbb{Z}_p^\times$. Unlike in ordinary algebra, in homotopy theory, profinite Galois extension may fail to be faithful – a form of hyperdescent. In particular, it is not immediate that $R \xrightarrow{\sim} R[\omega_{p^\infty}^{(n)}]^{h\mathbb{Z}_p^\times}$.

There is an (a priori) further condition on R, called being $\mathrm{K}(n)$-local, meaning that R is not only $\mathrm{T}(n)$-local but also local with respect to the Lubin–Tate spectrum $\mathrm{E}(L, \mathbf{G})$. For such rings, we can show that the infinite cyclotomic extension is faithful, and we prove the following result.

Theorem 7.2 (B.M.–Carmeli–Schlank–Yanovski)**.** *For a $\mathrm{T}(n)$-local commutative ring spectrum R we have*

$$L_{\mathrm{K}(n+1)}\mathrm{K}(R) \simeq L_{\mathrm{K}(n+1)}\mathrm{K}(R[\omega_{p^\infty}^{(n)}])^{h\mathbb{Z}_p^\times}.$$

Particularly, this provides an instance of hyperdescent for algebraic K-theory.

Finally, in the 70's, Ravenel posed the telescope conjecture, stating that any $\mathrm{T}(n)$-local spectrum is already $\mathrm{K}(n)$-local. This was finally resolved in the negative, as Burklund–Hahn–Levy–Schlank gave a counter example to the analogue theorem of ours for $\mathrm{T}(n+1)$-localized spectra.