Complex oriented rotation invariance in algebraic K-theory

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Plan

- Commutativity and strict commutativity
- Picard and strict Picard
- Units in algebraic K-theory
- Strict triviality in algebraic K-theory

Commutativity in ordinary algebra

- For elements of an abelian group $x, y \in A$

$$x \cdot y = y \cdot x$$

- In particular, for $x = y$

$$x \cdot x = x \cdot x$$

nothing much to say

- For objects in a category $X, Y \in \mathcal{C}$

$$B_{X,Y}\colon X \times Y \simeq Y \times X, \qquad B_{X,Y}(x,y) = (y,x)$$

- In particular, for $X = Y$

$$B_{X,X}\colon X \times X \simeq X \times X, \qquad B_{X,X}(x_1,x_2) = (x_2,x_1)$$

- More generally, Σ_n acts on X^n

Commutativity in higher algebra (cont.)

- Given an abelian ∞-group, i.e. a connective spectrum $(A, \otimes)$, for any $X \in A$ have

$$\Sigma_n \curvearrowright X^{\otimes n} \qquad \in \qquad A$$

- Element $X \in A \qquad \Longleftrightarrow \qquad$ map of spectra $\mathbb{S} \to A$

  encoding all Σ_n-action via

$$\mathbb{S} \simeq \mathrm{Fin}^{\mathrm{gpc}} \simeq \left(\bigsqcup \mathrm{B}\Sigma_n\right)^{\mathrm{gpc}}$$

Strict commutativity

- To say that $X \in A$ "commutes with itself" is extra structure: trivialization of $\Sigma_n \curvearrowright X^{\otimes n}$

Definition

The structure of a *strict element* on $X \in A$ is a factorization

$$\begin{tikzcd} \mathbb{S} \arrow[r, "X"] \arrow[d] & A \\ \mathbb{Z} \arrow[ur, dashed, bend right] & \end{tikzcd} \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

These assemble into a connective spectrum $A_{\mathbb{Z}} := \hom(\mathbb{Z}, A)$

Strict commutativity (cont.)

- Distinction does not exist in ordinary algebra since $\mathbb{Z}$ is both the free group and the free abelian group
- $\mathbb{Z}$ is the free ∞-group, i.e. $\mathbb{E}_1$-group, while $\mathbb{S}$ is the free abelian ∞-group, i.e. connective spectrum
- Element $X \in A \quad \Longleftrightarrow \quad \mathbb{E}_1$-map $\mathbb{Z} \to A$, and a strict structure is an $\mathbb{E}_\infty$-lift

Example

The neutral element $\mathbf{1} \in A$ canonically lifts to a strict element

Picard

- For a commutative ring spectrum R, consider the Picard

$$\mathrm{pic}(R) := \mathrm{Mod}_R(\mathrm{Sp})^{\simeq,\times},$$

maximal groupoid on X such that $\exists Y$ with $X \otimes Y \simeq R$, forming a connective spectrum with respect to $\otimes$

- Glued from

$$\pi_0(\mathrm{pic}(R)) \simeq \{X \mid \exists Y : X \otimes Y \simeq R\}, \qquad \tau_{\geq 1}(\mathrm{pic}(R)) \simeq R^\times$$

- Note for ordinary R, we also have $\Sigma^n R \in \pi_0(\mathrm{pic}(R))$

$$\pi_0(\mathrm{pic}(R)) \simeq \mathrm{pic}^\heartsuit(R) \oplus \mathbb{Z}^{\mathrm{Spec}(R)}$$

Picard (cont.)

Example ($R = \mathbb{Z}$)

$$\pi_0(\mathrm{pic}(\mathbb{Z})) = \{\Sigma^n \mathbb{Z}\} = \mathbb{Z}$$

$$\pi_1(\mathrm{pic}(\mathbb{Z})) \simeq \mathbb{Z}^\times \simeq \mathbb{Z}/2$$

$$\begin{tikzcd}
\mathrm{pic}(\mathbb{Z}) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & 0 \arrow[d] \\
\mathbb{Z} \arrow[r, "/2"] & \mathbb{Z}/2 \arrow[r, "\mathrm{Sq}^2"] & \Sigma^2 \mathbb{Z}/2
\end{tikzcd}$$

Strict Picard

Proposition

For an ordinary commutative ring, any ordinary Picard element $X \in \mathrm{pic}^{\heartsuit}(R)$ lifts to a strict Picard element $X \in \mathrm{pic}(R)_{\mathbb{Z}}$

Proof.

Locally on $\mathrm{Spec}(R)$, it is the unit $\square$

Strict Picard (cont.)

Proposition

For an ordinary commutative ring, $\Sigma^n R \in \mathrm{pic}(R)$ lifts to a strict Picard element if n is even, and doesn't (in general) if n is odd

Proof.

Suffices $R = \mathbb{Z}$, want to factor $\mathbb{S} \to \mathrm{pic}(\mathbb{Z})$ through $\mathbb{S} \to \mathbb{Z}$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[dr, dashed] \arrow[drrr, bend left] \arrow[ddr, bend right, "n"'] & & & \\
& \mathrm{pic}(\mathbb{Z}) \arrow[rr] \arrow[d] \arrow[drr, phantom, very near start, "\lrcorner"] & & 0 \arrow[d] \\
& \mathbb{Z} \arrow[r, "/2"] & \mathbb{Z}/2 \arrow[r, "\mathrm{Sq}^2"] & \Sigma^2\mathbb{Z}/2
\end{tikzcd}
$$

□

Picard of commutative ring spectra

Proposition

$\pi_0(\mathrm{pic}(\mathbb{S})) = \{\Sigma^n \mathbb{S}\} \simeq \mathbb{Z}$

- $\Sigma\mathbb{S}$ is not strict, by the case of $\mathbb{Z}$

Theorem (Lurie (2015))

There is an $\mathbb{E}_2$-lift of $\Sigma^2\mathbb{S}$

$$\mathbb{Z} \longrightarrow \mathrm{pic}(\mathbb{S}) \qquad \in \qquad \mathrm{Grp}_{\mathbb{E}_2}(\mathcal{S})$$

- No $\mathbb{E}_3$-lift, in particular, no strict lift

Picard of commutative ring spectra (cont.)

- Since $\mathrm{Aut}_{\mathrm{Cat}_{\mathrm{st}}}(\mathrm{Sp}^\omega) \simeq \mathrm{pic}(\mathbb{S})$, the $\mathbb{E}_2$-map

$$\mathbb{Z} \longrightarrow \mathrm{pic}(\mathbb{S}) \qquad \in \qquad \mathrm{Grp}_{\mathbb{E}_2}(\mathcal{S})$$

  deloops to an action of $S^1 \simeq \mathrm{B}\mathbb{Z}$ on $\mathrm{Cat}_{\mathrm{st}}$

- Lurie gives two constructions of the $\mathbb{E}_2$-lift
  - A "combinatorial" construction, via an analysis of filtrations (at the core, exhibiting as endomorphisms of the unit of a certain monoidal category)
  - A geometric construction, via the J-homomorphism

$$j\colon \mathrm{ku} \longrightarrow \mathrm{pic}(\mathbb{S}) \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

    which sends $1 \in \mathbb{Z} \simeq \pi_0(\mathrm{ku})$ to $\Sigma^2\mathbb{S}$, using Bott periodicity

Recap on commutativity and Picard

- Elements in a connective spectrum $X \in A$ have non-trivial

$$\Sigma_n \curvearrowright X^{\otimes n} \qquad \in \qquad A$$

- Trivialization is extra data

$$A_{\mathbb{Z}} := \hom(\mathbb{Z}, A) \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

- Ordinary Picard elements are always strict
- For ordinary rings $\Sigma^2 R \in \mathrm{pic}(R)_{\mathbb{Z}}$ is strict, but ΣR isn't
- $\Sigma^2 \mathbb{S} \in \mathrm{pic}(\mathbb{S})$ has an $\mathbb{E}_2$-lift

Algebraic K-theory

- For a commutative ring spectrum R consider

$$\mathrm{Perf}(R) := \mathrm{Mod}_R(\mathrm{Sp})^\omega \qquad \in \qquad \mathrm{CAlg}(\mathrm{Cat}_{\mathrm{st}})$$

- $\mathrm{K}(R)$ obtained from $\mathrm{Perf}(R)^\simeq$ by splitting exact sequences

$$X \longrightarrow Y \longrightarrow Z \qquad \rightsquigarrow \qquad [X] + [Z] \simeq [Y]$$

  implemented by $S_\bullet$-construction
- $\mathrm{K}(R) \in \mathrm{CAlg}(\mathrm{Sp}_{\geq 0})$ is a commutative ring spectrum with

$$[X] + [Y] = [X \oplus Y], \qquad [X] \cdot [Y] = [X \otimes Y]$$

Units in K-theory

- By construction, map of commutative ring spectra

$$\begin{aligned} \mathrm{Perf}(R)^{\simeq} &\longrightarrow \mathrm{K}(R) \qquad \in \qquad \mathrm{CAlg}(\mathrm{Sp}_{\geq 0}) \\ X &\longmapsto [X] \end{aligned}$$

- Multiplicatively invertible elements

$$\mathrm{pic}(R) := (\mathrm{Perf}(R)^{\simeq})^{\times} \longrightarrow \mathrm{K}(R)^{\times} \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

Example

$\Sigma^n R \in \mathrm{pic}(R)$ gives $[\Sigma^n R] \in \mathrm{K}(R)^{\times}$,
if R is ordinary, $[\Sigma^2 R] \in \mathrm{K}(R)^{\times}_{\mathbb{Z}}$ is strict

Splitting of units

Theorem (Carmeli–Luecke (2024))

The second k-invariant of $\mathrm{K}(\mathbb{Z})^\times$ vanishes, i.e.

$$\mathrm{K}(\mathbb{Z})^\times \simeq \tau_{\leq 1}\,\mathrm{K}(\mathbb{Z})^\times \oplus \tau_{\geq 2}\,\mathrm{K}(\mathbb{Z})^\times \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

(same for $\mathrm{K}(-)^\times$ of ordinary rings, after Zariski sheafification)

- Sophisticated analysis using chromatic methods
- Full computation of the strict units $\mathrm{K}(\mathbb{Z})^\times_{\mathbb{Z}}$

Alternative approach

Problem

Is $[\Sigma^2\mathbb{Z}] \in \mathrm{K}(\mathbb{Z})^\times_{\mathbb{Z}}$ trivial? I.e., is the following null-homotopic

$$\mathbb{Z} \longrightarrow \mathrm{pic}(\mathbb{Z}) \longrightarrow \mathrm{K}(\mathbb{Z})^\times \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

Alternative proof of splitting.

If true, would give a factorization

$$\begin{tikzcd}
\mathrm{pic}(\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{K}(\mathbb{Z})^\times \arrow[r] & \tau_{\leq 1}\,\mathrm{K}(\mathbb{Z})^\times \\
\mathrm{pic}(\mathbb{Z})/2\mathbb{Z} \arrow[ur, dashed, bend right] & &
\end{tikzcd}$$

and the composition is an isomorphism, providing a section $\square$

Question over complex cobordism

- Complex cobordism MU is the Thom spectrum of

$$\mathrm{bu} = \tau_{\geq 2}\mathrm{ku} \longrightarrow \mathrm{ku} \xrightarrow{j} \mathrm{pic}(\mathbb{S}) \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

i.e., equipped with a null-homotopy of

$$\mathrm{bu} \longrightarrow \mathrm{pic}(\mathbb{S}) \longrightarrow \mathrm{pic}(\mathrm{MU}) \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

hence receives a map

$$\mathbb{Z} \simeq \mathrm{ku}/\mathrm{bu} \longrightarrow \mathrm{pic}(\mathrm{MU}) \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

- Canonically get $\Sigma^2\mathrm{MU} \in \mathrm{pic}(\mathrm{MU})_{\mathbb{Z}}$

Complex oriented strict triviality

Theorem (B.-M., in progress)

For any map of commutative ring spectra $\mathrm{MU} \to R$*, K-theory trivializes* $\Sigma^2 R \in \mathrm{pic}(R)_{\mathbb{Z}}$*, i.e., the composition*

$$\mathbb{Z} \longrightarrow \mathrm{pic}(\mathrm{MU}) \longrightarrow \mathrm{pic}(R) \longrightarrow \mathrm{K}(R)^{\times} \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

is null-homotopic

Example

For any ordinary commutative ring R have

$$\mathrm{MU} \longrightarrow \tau_{\leq 0}\mathrm{MU} \simeq \mathbb{Z} \longrightarrow R$$

Triviality of $\Sigma^2 R$

- ΣX participates in

$$X \longrightarrow 0 \longrightarrow \Sigma X \qquad \rightsquigarrow \qquad [\Sigma X] \simeq -[X]$$

- Applied twice to $X = R$

$$[\Sigma^2 R] \simeq [R] \qquad \in \qquad \mathrm{K}(R)^\times$$

- That is, the map

$$\mathbb{S} \longrightarrow \mathrm{pic}(R) \longrightarrow \mathrm{K}(R)^\times \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

choosing $[\Sigma^2 R]$ is null-homotopic

Rotation invariance

Theorem (Lurie (2015))

K-theory trivializes the $\mathbb{E}_2$-lift of $\Sigma^2\mathbb{S}$, i.e., the composition

$$\mathbb{Z} \longrightarrow \mathrm{pic}(\mathbb{S}) \longrightarrow \mathrm{K}(\mathbb{S})^\times \qquad \in \qquad \mathrm{Grp}_{\mathbb{E}_2}(\mathcal{S})$$

is null-homotopic

- Using the "combinatorial", rather than the geometric, $\mathbb{E}_2$-lift
- Lift corresponds to an S^1-action on $\mathrm{Cat}_{\mathrm{st}}$, trivialization corresponds to S^1-invariance of $\mathrm{K}\colon \mathrm{Cat}_{\mathrm{st}} \to \mathrm{Sp}$
- Present K-theory via the *paracyclic* $S_\bullet$-construction, extra functoriality makes it S^1-equivariant

Complex oriented rotation invariance

Theorem (B.-M., in progress)

K-theory trivializes $\Sigma^2\mathrm{MU} \in \mathrm{pic}(\mathrm{MU})_{\mathbb{Z}}$*, i.e., the composition*

$$\mathbb{Z} \longrightarrow \mathrm{pic}(\mathrm{MU}) \longrightarrow \mathrm{K}(\mathrm{MU})^{\times} \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

is null-homotopic

- Lurie's identification transfers from geometric side (J-homomorphism) to "combinatorial" side
- Giving a strict S^1-action on $\mathrm{Perf}(\mathrm{MU})$-linear categories
- Make the paracyclic $S_\bullet$-construction of $\mathrm{Perf}(\mathrm{MU})$-linear categories strictly S^1-equivariant

Recap

- $\Sigma^2 R \in \mathrm{pic}(R)$ strictly commutes with itself for ordinary R
- Doesn't over $\mathbb{S}$, but does already over MU
- Splitting of exact sequence in K-theory trivializes $\Sigma^2 R$
- And its strict structure

Thank You!