Six Functor Formalisms Seminar
Duality for Lie groups and J-Homomorphisms

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22/01/2026

0 Overview

This talk covers Sections 10 and 11 of Dustin's paper arXiv:2506.18174. We are going to apply all the technology developed in the last few talks to prove the main theorem of the paper. I'd like to start by stating the main theorem soon, even without defining everything, to ground the discussion. To that end, let's first recall that Atiyah duality connects the cohomology and homology of a smooth manifold, which I'd like to state in two parts.

Theorem 0.1 (Atiyah duality)**.** *Let M be a smooth manifold and denote $f\colon M \to \mathrm{pt}$.*

(1) f is Sp-smooth, that is, for any spectrum E, the map

$$f^*E \otimes \omega_f \xrightarrow{\sim} f^!E, \qquad \omega_f := f^!\mathbb{S}$$

is an isomorphism, and ω_f is invertible.

(2) There is a canonical isomorphism $\omega_f \simeq \mathbb{S}^{TM}$.

Example 0.2 (Poincare duality)**.** Assume that M is compact, so that f is Sp-proper. Taking $E = \mathbb{Z}$ and applying $f_!$ we recover Poincare duality in the following form

$$C^\bullet(M;\mathbb{Z})^\vee \simeq C^\bullet(M;\Lambda^d\mathbb{Z}^{TM}[d]),$$

where $\Lambda^d\mathbb{Z}^{TM}$ is the orientation local system.

We want to prove a variant of this theorem for real and p-adic Lie groups, namely for $f\colon \mathrm{B}G \to \mathrm{pt}$ with $G \in \mathrm{Grp}(\mathrm{Man}_F)$. With integral (rather than spectral) coefficients this was done by Lazard(–Serre). In the previous talk, we have proven (or at least stated) Sp_p-smoothness, which essentially reduces to Lazard's case. In light of this, the real content, which is the main theorem of the paper, is the identification of the dualizing object ω_f, to which we will dedicate the first half of the lecture.

Theorem 0.3. *Let G be a real Lie group, and let $f\colon \mathrm{B}G \to \mathrm{pt}$, then*

$$\omega_f = f^!\mathbb{S} \simeq \mathbb{S}^{-\mathrm{ad}_G} \qquad \in \qquad \mathrm{Sh}_{\mathrm{et}}(\underline{\mathrm{B}G}; \mathrm{Sp}),$$

where $\mathrm{ad}_G\colon V \to \mathrm{B}G$ is the adjoint representation viewed as a vector bundle, and $\underline{\mathrm{B}G}$ is the associated condensed anima.

If G is instead a p-adic Lie group, working with p-adic coefficients, we have

$$\omega_f = f^!\mathbb{S}_p \simeq \mathbb{S}_p^{\mathrm{ad}_G} \qquad \in \qquad \mathrm{Sh}_{\mathrm{et}}(\underline{\mathrm{B}G}; \mathrm{Sp}_p),$$

The lack of a minus sign in the latter is not a typo, but it is a normalization choice which I will explain later. Of course, I also haven't told you yet what is $\mathbb{S}^V$.

Example 0.4. Let G be a p-adic Lie group. We have shown last time that if it is compact and p-torsion-free, then $f\colon \mathrm{B}G \to \mathrm{pt}$ is also Sp_p-proper. One can check that after base-change to $\mathbb{Z}_p$ the dualizing object simplifies to something analogues to the orientation local system above, and we recover Lazard duality

$$C^\bullet(G; \mathbb{Z}_p)^\vee \simeq C^\bullet(G; (\Lambda^d \mathrm{ad}_G)_{\mathbb{Z}_p}[d]).$$

In the second half of the lecture we will carry out a more systematic study of $V \mapsto \mathbb{S}^V$. As is clear from the notation, this is closely related to the J-homomorphism, giving in particular a p-adic analogue. In fact, we will also prove a reciprocity law for them, which implies Artin reciprocity law from class field theory. Thus the plan for today is:

(1) A uniform proof of Atiyah duality and Lie group dualities.

(2) J-homomorphisms and the reciprocity law.

(3) Bonus: Artin reciprocity.

1 Atiyah duality and linearization of Lie groups

1.1 Deformation to the tangent bundle

Our first goal is proving Atiyah duality for representable submersions. Let us begin by recalling some definitions from Daniel Arone's talk. For $X \in \mathrm{Man}_F$, we have defined another manifold TX equipped with a projection and a zero section. Importantly, we also defined the deformation space, which comes with a projection $\Pi\colon DX \to F \times X$ and a section Σ, which are $F^\times$-equivariant. The main features of this constructions are that

$$DX \times_F \{0\} \simeq TX, \qquad DX \times_F F^\times \simeq F^\times \times X \times X,$$

where over 0, Π is the projection and Σ is the zero section, and away from 0 they are projection to the first factor and the diagonal. Namely, we have an ($F^\times$-equivariant) deformation of $X \times X$ to TX.

These constructions also relativize: for a submersion $f\colon X \to Y$ in Man_F, we get a relative tangent bundle $T(X/Y) = \ker(TX \to f^*TY)$ and deformation space $D(X/Y)$ satisfying exactly the same properties. Since T and D are sheaves (i.e. local on the target), we can generalize to any representable submersion $f\colon X \to Y$ in $\mathrm{Sh}(\mathrm{Man}_F)$.

1.2 Thom objects

In the statement of Atiyah duality we would like to have $\mathbb{S}^{T(X/Y)}$, so we better define this. We start with the real case.

Definition 1.1. Let $X \in \mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$ and let $f\colon V \to X$ be a vector bundle with zero section $e\colon X \to V$. We define

$$\mathbb{S}^V := e^* f^! \mathbb{S} \qquad \in \qquad \mathrm{Sh}_{\mathrm{et}}(\underline{X}; \mathrm{Sp}).$$

By Sp-smoothness of f, we learn that $f^!\mathbb{S}$ is an invertible object, and thus so is $\mathbb{S}^V$. Hence, it is locally constant, thus lifting to the subcategory

$$\mathrm{An}_{/|X|} \otimes \mathrm{Sp} \simeq \mathrm{Sh}_{\mathrm{et}}(|X|; \mathrm{Sp}) \hookrightarrow \mathrm{Sh}_{\mathrm{et}}(\underline{X}; \mathrm{Sp}).$$

Remark 1.2. This admits a more familiar description in terms of the complement of the zero section $V \setminus 0 \to V$, by a standard six functor formalism argument

$$\mathbb{S}^V \simeq \Sigma^\infty(|V| \,/\, |V \setminus 0|)$$

where here everything is over $|X|$.

1.3 Atiyah duality

We are now in position to prove Atiyah duality for a relative submersion $f\colon X \to Y$ in $\mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$. Recall that the associated map of condensed anima $f\colon \underline{X} \to \underline{Y}$ is Sp-smooth, by reducing to manifolds, so it remains to identify the dualizing object.

Theorem 1.3. *Let $f\colon X \to Y$ be a representable submersion in* $\mathrm{Sh}(\mathrm{Man}_F)$*, then*

$$\omega_f := f^!\mathbb{S} \simeq \mathbb{S}^{T(X/Y)} \qquad \in \qquad \mathrm{Sh}_{\mathrm{et}}(\underline{X}; \mathrm{Sp}).$$

The idea of the proof will be to use the deformation space $D(X/Y)$ to interpolate between the fiber of 1 and the fiber of 0. Thus, before proceeding with the proof, let me give the following very general and easy formula.

Lemma 1.4. *Let $f\colon X \to Y$ be a smooth map for some six functor formalism. Then*

$$\omega_f = f^!\mathbf{1} \simeq \delta^*\pi_1^!\mathbf{1} \qquad \in \qquad D(X)$$

where the maps are the diagonal $\delta\colon X \to X \times_Y X$ and first projection $\pi_1\colon X \times_Y X \to X$.

Proof. Indeed, using base-change for upper shriek, we get

$$f^! \simeq \mathrm{Id}^* f^! \simeq \delta^*\pi_2^* f^! \simeq \delta^*\pi_1^! f^* \simeq \delta^*\pi_1^!.$$

□

Proof of Atiyah duality. Consider the deformation space, and recall that $\Pi\colon D(X/Y)\to \mathbb{R}\times X$ is a representable submersion with section Σ. Arguing as for $\mathbb{S}^V$ above, smoothness implies that $\mathbb{S}^{D(X/Y)} := \Sigma^*\Pi^!\mathbb{S} \in \mathrm{Sh}_{\mathrm{et}}(\underline{\mathbb{R}\times X}, \mathrm{Sp})$ is locally constant. Hence it comes from the full subcategory $\mathrm{Sh}_{\mathrm{et}}(|\mathbb{R}\times X|;\mathrm{Sp}) \simeq \mathrm{An}_{/|\mathbb{R}\times X|}\otimes \mathrm{Sp}$, and since $\mathbb{R}$ is contractible, the fibers over 0 and 1 are the same

$$0^*\mathbb{S}^{D(X/Y)} \simeq 1^*\mathbb{S}^{D(X/Y)}.$$

The fiber of $D(X/Y)$ is $T(X/Y)$ (with the projection and zero section), and over 1 it is $X\times_Y X$ (with the first projection and diagonal). Using base-change and the diagonal lemma we get

$$\omega_f = f^!\mathbb{S} \simeq \delta^*\pi_1^!\mathbb{S} = \mathbb{S}^{X\times_Y X} \simeq 1^*\mathbb{S}^{D(X/Y)} \simeq 0^*\mathbb{S}^{D(X/Y)} \simeq \mathbb{S}^{T(X/Y)},$$

concluding the proof. □

1.4 Linearization of real Lie groups

We now wish to show that for $f\colon \mathrm{B}G\to \mathrm{pt}$ given a real Lie group $G\in \mathrm{Grp}(\mathrm{Man}_{\mathbb{R}})$, we have $\omega_f \simeq \mathbb{S}^{-\mathrm{ad}_G}$ in $\mathrm{Sh}_{\mathrm{et}}(\underline{\mathrm{B}G};\mathrm{Sp})$, where $\mathrm{ad}_G\to \mathrm{B}G$ is the Lie algebra $\mathfrak{g} := TG$ with the conjugation action, and the minus sign denotes dual (recall it is an invertible object). The argument is very similar to the above. We proceed in two steps: proving a version in families but forgetting the G-action, and deducing the result.

Proposition 1.5. *Let $X\in \mathrm{Sh}(\mathrm{Man}_{\mathbb{R}})$, and let $G\to X$ be a Lie group, i.e. a group object in representable submersions over X. Let $f\colon \mathrm{B}G\to X$ denote the relative classifying stack, and let $e\colon X\to \mathrm{B}G$ denote the relative basepoint. Let $\mathfrak{g} := T(G/X)\to X$ denote the Lie group viewed as a vector bundle. Then*

$$e^*\omega_f = e^*f^!\mathbb{S} \simeq \mathbb{S}^{-\mathfrak{g}} \qquad \in \qquad \mathrm{Sh}_{\mathrm{et}}(\underline{X};\mathrm{Sp}).$$

Proof. This is almost exactly as above, but rather than looking at $D(\mathrm{B}G/X)\to X$ we look at $D(G/X)\to X$, which is still a group object, and pass to its relative classifying stack. Note that the fibers this time are $\mathrm{B}\mathfrak{g}$ over 0 and the diagonal $\mathrm{B}G\times_X \mathrm{B}G$ over 1. Therefore, all that remains is to show that the Thom $\mathbb{S}^{\mathrm{B}\mathfrak{g}}$ of $\mathrm{B}\mathfrak{g}$ is dual to $\mathbb{S}^{\mathfrak{g}}$, which we prove in the following lemma. □

Lemma 1.6. *Let $X \xrightarrow{e} V \xrightarrow{f} X$ be a vector bundle, and denote $X \xrightarrow{e_{\mathrm{B}}} \mathrm{B}V \xrightarrow{f_{\mathrm{B}}} X$. Then $\mathbb{S}^{\mathrm{B}V} := e_{\mathrm{B}}^* f_{\mathrm{B}}^!\mathbb{S}$ is dual to $\mathbb{S}^V$.*

Proof. Note that e_{B} is smooth thus

$$\mathbb{S} \simeq e_{\mathrm{B}}^! f_{\mathrm{B}}^!\mathbb{S} \simeq e_{\mathrm{B}}^* f_{\mathrm{B}}^!\mathbb{S}\otimes e_{\mathrm{B}}^!\mathbb{S},$$

so it remains to show that $e_{\mathrm{B}}^!\mathbb{S}\simeq \mathbb{S}^V$. This follows from the diagonal lemma for the morphism e_{B}, noting that in this case $V\simeq X\times_{\mathrm{B}V} X$, the projection is f and the diagonal is e, so that $e_{\mathrm{B}}^!\mathbb{S}\simeq e^*f^!\mathbb{S} = \mathbb{S}^V$. □

We can now prove the theorem, recording the G-action.

Theorem 1.7. *Let G be a real Lie group, and let $f\colon \mathrm{B}G \to \mathrm{pt}$, then*

$$\omega_f := f^!\mathbb{S} \simeq \mathbb{S}^{-\mathrm{ad}_G} \qquad \in \qquad \mathrm{Sh}_{\mathrm{et}}(\underline{\mathrm{B}G};\mathrm{Sp}).$$

Proof. This follows immediately from the previous proposition in families, once we understand what exactly is ad_G, so let's do that. For a discrete group G, we have the functor

$$G^{\mathrm{ad}}\colon \mathrm{B}G \longrightarrow \mathrm{Grp}$$

sending the base-point to $\mathrm{pt} \in \mathrm{B}G$ to G with its G-action by conjugation. We can post-compose to get

$$\mathrm{B}G \xrightarrow{G^{\mathrm{ad}}} \mathrm{Grp} \xrightarrow{\sim} \mathrm{Grpd}_{\mathrm{pt,cn}} \longrightarrow \mathrm{Grpd},$$

and it is easy to see that this is the constant functor with value $\mathrm{B}G$. Taking the unstraightening, this is $\pi_1\colon \mathrm{B}G \times \mathrm{B}G \to \mathrm{B}G$. Functoriality in G shows that this passes to topoi, and thus this holds for our Lie group. To summarize, the projection encodes the adjoint action on G.

Consider then the group $\mathrm{B}G \times G$ as a group object relative to $\mathrm{B}G$, whose tangent bundle is then $T(BG \times G/BG) \simeq \mathrm{ad}_G$. The relative classifying stack is $\pi_1\colon \mathrm{B}G \times \mathrm{B}G \to \mathrm{B}G$ with basepoint $\delta\colon \mathrm{B}G \to \mathrm{B}G \times \mathrm{B}G$, we conclude using the diagonal lemma and the previous proposition

$$\omega_f = f^!\mathbb{S} \simeq \delta^*\pi_1^!\mathbb{S} \simeq \mathbb{S}^{-T(BG\times G/BG)} \simeq \mathbb{S}^{-\mathrm{ad}_G} \qquad \mathrm{Sh}_{\mathrm{et}}(\underline{\mathrm{B}G};\mathrm{Sp}).$$

□

1.5 Linearization of p-adic Lie groups

Stating the p-adic case requires modifying our definitions. Notably, $V \to X$ is generally *not* Sp-smooth: indeed, $\mathbb{Q}_p \to \mathrm{pt}$ isn't. However, $\mathrm{B}V \to X$ is Sp_p-smooth, as we have shown last time. Recall that we have seen above that $\mathbb{S}^{-\mathrm{B}V} \simeq \mathbb{S}^V$. In light of this we make the following definition in the p-adic case.

Definition 1.8. Let $X \in \mathrm{Sh}(\mathrm{Man}_{\mathbb{Q}_p})$ and let $V \to X$ be a vector bundle, and consider $X \xrightarrow{e_{\mathrm{B}}} \mathrm{B}V \xrightarrow{f_{\mathrm{B}}} X$. We define

$$\mathbb{S}_p^V := e_{\mathrm{B}}^* f_{\mathrm{B}}^! \mathbb{S} \qquad \in \qquad \mathrm{Sh}_{\mathrm{et}}(\underline{X};\mathrm{Sp}_p).$$

Remark 1.9. Note that we **do not introduce a minus sign** in the notation, inconsistently with the real case. This may seem very odd and confusing convention, but I can offer the following two reasons:

(1) In the real case, $\mathbb{S}^V$ of course lives in $\mathbb{Q}$-homology degree $\dim(V)$. Similarly, in the proof of the Sp_p-smoothness of $\mathrm{B}V \to X$, with the convention above, $\mathbb{S}_p^V$ lives in $\mathbb{F}_p$-homology degree $\dim(V)$.

(2) As we shall explain in the second half of the talk, this makes the statement of the reciprocity theorem more uniform, and in fact we don't really have room for choice.

With this in place, and being careful with the minus sign, we prove duality.

Theorem 1.10. *Let G be a p-adic Lie group, and let $f\colon \mathrm{B}G \to \mathrm{pt}$, then*

$$\omega_f := f^!\mathbb{S}_p \simeq \mathbb{S}_p^{\mathrm{ad}_G} \qquad \in \qquad \mathrm{Sh}_{\mathrm{et}}(\underline{\mathrm{B}G}; \mathrm{Sp}_p).$$

Proof. The proof goes in verbatim as in the real case, with one slight modification. Previously we argued that the 0 and 1 fibers agree because $\mathbb{R}$ is contractible. Now we have $\mathbb{Q}_p$ in place of $\mathbb{R}$, which is not contractible. Nevertheless, the construction is $\mathbb{Q}_p^\times$-equivariant, so the fibers still agree by the non-standard interval we have seen in the end of the last talk. ☐

2 J-homomorphisms and reciprocity

In the previous part of the talk, we have considered the construction $V \mapsto \mathbb{S}^V$. In this part, we would like to study this construction in more detail, as we vary V (but fix the base X). For example, we saw that $\mathbb{S}^{\mathrm{B}V}$ is dual to $\mathbb{S}^V$, and more generally, it is true that for a short exact sequence $V \to W \to W/V$ we have $\mathbb{S}^W \simeq \mathbb{S}^V \otimes \mathbb{S}^{V/W}$. We thus see that this is K-theoretic. Of course, as the notation suggest, this construction is reminiscent of the well-known J-homomorphism

$$J\colon \mathrm{ko} \longrightarrow \mathrm{Pic}(\mathbb{S}), \qquad J(V) = \mathbb{S}^V.$$

We are going to see that the construction $V \mapsto \mathbb{S}^V$ from the previous section assembles into a generalization of the J-homomorphism. They will be more general in two ways. First, we get an adic version, namely for F of characteristic $\neq p$ we get

$$J_F\colon K(F) \longrightarrow \mathrm{Pic}(\mathbb{S}_p).$$

Second, this will actually be defined for much "larger" objects e.g. locally compact vector spaces rather than just finite dimensional. We will also show that as we vary F, there is a certain product formula. These results were obtained by Dustin a previous paper arXiv:1110.5851. In another paper arXiv:1703.07842 he used this to give a new, homotopy theoretic construction of the Artin homomorphism – one of the key results of class field theory.

2.1 Vectorial objects

Much of the discussion can be axiomatized as follows. An interesting observation is that if $f\colon V \to X$ real vector bundle, then f^* is fully faithful (well known). This is not true in general for p-adic vector bundles, but does holds for the relative classifying stack $f\colon \mathrm{B}V \to X$ (as we have essentially seen in the previous talk). Passing to relative classifying stack one more time, something even better happens: f^* is an equivalence, with an inverse f_*; by smoothness $f^!$ is also an equivalence with inverse $f_!$. As we'll see soon, using this as our definition, we can give a very clean treatment. To that end, we fix a six functor formalism, i.e. a lax symmetric monoidal functor $D\colon \mathrm{Span}_E(\mathcal{C}) \to \mathrm{Pr}^{\mathrm{L}}$.

Definition 2.1. We say that a map $f\colon X \to Y$ in $\mathcal{C}$ is vectorial if it is in E and for every base-change $\widetilde{f}$ of f, the functors $\widetilde{f}^*$ and $\widetilde{f}_!$ are equivalences. We say that an object $X \in \mathcal{C}$ is vectorial if $f\colon X \to \mathrm{pt}$ is.

Proposition 2.2. *Vectorial implies smooth.*

Remark 2.3. $f\colon X \to Y$ is vectorial if and only if $X \in \mathcal{C}_{/Y}$ is, so we can often work with objects.

Definition 2.4. For a vectorial object $X \in \mathcal{C}$, letting $f\colon X \to \mathrm{pt}$, we define $J(X)$ to be the compactly supported cohomology

$$J(X) := f_! f^* \mathbf{1} \qquad \in \qquad D(\mathrm{pt}).$$

Since both functors are $D(\mathrm{pt})$-linear equivalences, it immediately follows that object is invertible. If we have a section, this also agrees with the definition from the previous section, up to inverting.

Proposition 2.5. *Assume X is vectorial, and we have a section $e\colon \mathrm{pt} \to X$, then*

$$J(X) := f_! f^* \mathbf{1} \simeq (e^* f^! \mathbf{1})^{-1} \qquad \in \qquad D(\mathrm{pt}).$$

Construction 2.6. We construct a map of connective spectra

$$J\colon K(\mathrm{Vect}_D) \longrightarrow \mathrm{Pic}(D(*)) \qquad \in \qquad \mathrm{Sp}_{\geq 0},$$

refining the construction $J(X)$ above.

We first describe $K(\mathrm{Vect}_D)$. Define $Q(\mathrm{Vect}_D) := \mathrm{Span}_V(\mathrm{Vect}_D) \subset \mathrm{Span}_E(\mathcal{C})$ to be the subcategory on spans where the objects are vectorial and the covariant morphisms are vectorial. This is symmetric monoidal via the cartesian structure on Vect_D. Take geometric realization and loop $K(\mathrm{Vect}_D) := \Omega|Q(\mathrm{Vect}_D)|$, which is a a group-like $\mathbb{E}_\infty$-space, i.e. a connective spectrum.

To construct J, start with the lax symmetric monoidal $D\colon \mathrm{Span}_E(\mathcal{C}) \to \mathrm{Mod}_{D(\mathrm{pt})}(\mathrm{Pr}^{\mathrm{L}})$. Restrict to $Q(\mathrm{Vect}_D)$. By vectoriality $D(X)$ is isomorphic to $D(\mathrm{pt})$ (via f^* or $f_!$, since $J(X)$ is invertible) and in particular lands in the invertible objects. By 2-out-of-3 any morphisms between vectorial objects is sent an invertible morphisms. Thus, the restriction gives $D\colon Q(\mathrm{Vect}_D) \to \mathrm{Pic}(\mathrm{Mod}_{D(\mathrm{pt})}(\mathrm{Pr}^{\mathrm{L}}))$. By adjunction, this factors through the geometric realization. Taking loop, we get $J\colon K(\mathrm{Vect}_D) \to \mathrm{Pic}(D(\mathrm{pt}))$.

It makes sense to extend the definitions above from objects of $\mathcal{C}$ to spectrum objects. One reason is that our central example – vector bundles – do have an additive structure. A second reason is in the p-adic case, $\mathbb{Q}_p \to \mathrm{pt}$ was not even smooth, but taking $\mathrm{B}^2\mathbb{Q}_p$ is vectorial, so we'd like to say that $\mathbb{Q}_p$ itself is vectorial if we record its additive structure. I'll be light on the details here in the interest of time.

Definition 2.7. We say that the six functor formalism $D\colon \mathrm{Span}_E(\mathcal{C}) \to \mathrm{Pr}^{\mathrm{L}}$ has good descent if $\mathcal{C}$ is an ∞-topos, D^* is limit preserving, and the condition for $f\colon X \to Y$ to be in E is local on Y.

Example 2.8. This holds for $\mathrm{Sh}_{\mathrm{et}}$ on condensed anima.

From now on we assume this condition.

Definition 2.9. We say that $X \in \mathcal{C} \otimes \mathrm{Sp}$ is vectorial if it is bounded below, and $\Omega^\infty \Sigma^n X \in \mathcal{C}$ is vectorial for n large enough. We denote $\mathrm{vect}_D \subset \mathcal{C} \otimes \mathrm{Sp}$ for their full subcategory.

Construction 2.10. We construct map of connective spectra

$$J\colon K(\mathrm{vect}_D) \longrightarrow \mathrm{Pic}(D(*)) \qquad \in \qquad \mathrm{Sp}_{\geq 0}$$

refining the previous map. This time, vect_D is a stable category, and by K-theory we really mean any of the normal K-theories, e.g. the Q-construction $K(\mathrm{vect}_D) := \Omega|\mathrm{Span}(\mathrm{vect}_D)|$. This is a refinement in the following more precise sense. We let $Q(\mathrm{vect}_{M,\geq 1}) \subset \mathrm{Span}(\mathrm{vect}_D)$ denote the subcategory where objects have $\pi_i X = 0$ for $i \leq 0$ and $\Omega^\infty X$ is vectorial, and morphisms $X \leftarrow C \to Y$ are such that $\pi_i C = 0$ for $i \leq 0$ and $C \to Y$ is surjective on π_1 and its Ω^∞ is vectorial. By construction this fits into

$$Q(\mathrm{Vect}_M) \xleftarrow{\Omega^\infty} Q(\mathrm{vect}_{M,\geq 1}) \hookrightarrow \mathrm{Span}(\mathrm{vect}_D).$$

One proves (and we won't) that the right map induces an isomorphism on $\Omega|-|$, allowing us to extend J along the left morphism compatibly.

2.2 Proper and etale triviality

Many of the objects we consider in practice are proper or etale. It turns out that these have trivial J. Indeed, say $f\colon X \to \mathrm{pt}$ is proper, i.e. $f_! \simeq f_*$. Vectoriality says that f_* is inverse to f^*, hence

$$J(X) := f_! f^* \mathbf{1} \simeq \mathbf{1}.$$

Let us prove this functorially way. Let $Q(\mathrm{Vect}_D^\pi), Q(\mathrm{Vect}_D^\epsilon) \subset Q(\mathrm{Vect}_D)$ denote the subcategories where the objects are proper (respectively etale) and so are the objects in the spans. We define their K-theory by $\Omega|-|$ as above.

Theorem 2.11. *There is a canonical trivialization of*

$$K(\mathrm{Vect}_D^\pi) \longrightarrow K(\mathrm{Vect}_D) \xrightarrow{J} \mathrm{Pic}(D(*))$$

and similarly for etale.

Proof. The etale case is dual, so we only discuss the proper case.

With out loss of generality we may assume that all objects of $\mathcal{C}$ are proper (which are closed under finite limits, hence the six functor formalism restricts). We want to use

Eilenberg swindle, so we need to have countable products. Let $\widetilde{\mathcal{C}} := \mathrm{Pro}(\mathcal{C})$, and we wish to extend the six functor formalism. Extending $D^*\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$ by filtered colimits, we get $\widetilde{D}^*\colon \widetilde{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$. By properness, for every morphism f in $\mathcal{C}$, the morphism f^* has a right adjoint $f_* \simeq f_!$ satisfying the projection formula and commuting with pullback. This is inherited to morphisms in $\widetilde{\mathcal{C}}$, since the formula for adjoint of a limit is the limit of the adjoints. Thus, taking $\widetilde{E} = \widetilde{\mathcal{C}}, \widetilde{P} = \widetilde{\mathcal{C}}$ and $\widetilde{I} = \mathrm{iso}$, by Heyer–Mann, we get a six functor formalism $\widetilde{D}\colon \mathrm{Span}(\widetilde{\mathcal{C}}) \to \mathrm{Pr}^{\mathrm{L}}$.

Consequently, we get a factorization of J as

$$K(\mathrm{Vect}_D) \longrightarrow K(\mathrm{Vect}_{\widetilde{D}}) \longrightarrow \mathrm{Pic}(D(\mathrm{pt})),$$

so it suffices to show that the middle term vanishes. Since $\mathrm{Vect}_{\widetilde{D}}$ has countable products, we finish bu Eilenberg swindle: Intuitively, for every object X we have $\prod_{\mathbb{N}} X \simeq (\prod_{\mathbb{N}} X) \times X$, so in K we get $[\prod_{\mathbb{N}} X] = [\prod_{\mathbb{N}} X] + [X]$ namely $[X] = 0$. Writing Id in place of X, this shows that $K(\mathrm{Vect}_{\widetilde{D}}) = 0$ as required. ☐

The entire discussion above follows for the spectrum objects version vect_D.

2.3 J-homomorphism for condensed anima

Back to reality, let's consider the six functor formalism $\mathrm{Sh}_{\mathrm{et}}(-;\mathrm{Sp}_p)$ for a fixed prime p. We give examples of vectorial, sometimes also proper or etale, spectrum objects M arising from condensed abelian groups. Recall that by definition M is vectorial as a spectrum object if and only if $\mathrm{B}^n M$ is for some n – in practice $n = 2$ will always suffice.

Theorem 2.12. *The following condensed abelian groups are vectorial for* $\mathrm{Sh}_{\mathrm{et}}(-;\mathrm{Sp}_p)$ *when considered as condensed spectra:*

(1) Any discrete $\mathbb{Z}[1/p]$-module, which is moreover etale.

(2) Any compact Hausdorff second countable and finite dimensional $\mathbb{Z}[1/p]$-module, which is moreover proper.

(3) For any local field F of characteristic $\neq p$, any finite dimensional F-vector space.

Proof. For 1, it suffices to show that $f\colon \mathrm{B}M \to \mathrm{pt}$ is etale and has f^* is fully faithful (which is inherited for any pullback and then pullback along $\mathrm{B}^2 M \to \mathrm{pt}$ is an equivalence; etaleness treats $f_!$). We have seen before that it is in fact Sp-etale. To check fully faithfulness of f^*, consider its *left* adjoint $f_\natural$, and we need to show that for any p-complete X, the counit $f_\natural f^* X \to X$ from the M-homology is an isomorphism. p-complete equivalences can be checked mod p, and by induction on the Postnikov tower, we reduce to $X = \mathbb{F}_p$. In this case, by resolving M and Kunneth, we reduce to $M = \mathbb{Z}[1/p]$, in which case it is an easy computation.

Case 2 is dealt in an essentially dual case.

For 3, we first handle non-archimedean F of residue characteristic $\neq p$. Consider the short exact sequence $\mathcal{O}_F \to F \to F/\mathcal{O}_F$. The first term is handled by 1 and the third by 2, and since vect is stable, we get that F is vectorial. Thus so is $M = \oplus_k F$. For archimedean we need to handle $\mathbb{R} \to \mathrm{pt}$, this follows by a standard argument from the fact that this map is smooth and pullback along it is fully faithful. Finally, for non-archimedean F of residue characteristic p, we are reduced to $\mathbb{Q}_p$. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Q}_p \oplus \mathbb{R} \longrightarrow C \longrightarrow 0.$$

The first term is vectorial by 1. The last term $C = \lim_n \mathbb{R}/p^n\mathbb{Z}$ is vectorial by 2. By stability $\mathbb{Q}_p \oplus \mathbb{R}$ is vectorial. Since we already know that $\mathbb{R}$ is vectorial, we get that $\mathbb{Q}_p$ is vectorial as required. ☐

Applying the general theory, we get the following.

Definition 2.13. We have a J-homomorphism

$$J\colon K(\mathrm{vect}_{\mathrm{Sh}_{\mathrm{et}}(-;\mathrm{Sp}_p)}) \longrightarrow \mathrm{Pic}(\mathbb{S}_p).$$

By the previous theorem, we can restrict to finite dimensional vector spaces over a local field F of characteristic $\neq p$, yielding

$$J_F\colon K(F) \longrightarrow \mathrm{Pic}(\mathbb{S}_p).$$

Remark 2.14. For $F = \mathbb{R}$ the proof shows that this refines to $\mathrm{Pic}(\mathbb{S})$. In the non-archimedean case, if the residue characteristic $\ell \neq p$ this similarly refines to $\mathrm{Pic}(\mathbb{S}[1/\ell])$.

2.4 Product formula

Our final theorem from this paper for today is the following (which one actually wants for infinite S).

Theorem 2.15. *Let F be a global field of characteristic $\neq p$, and S finite set of place containing all infinite places of F and those above p. Then there is a null-homotopy of*

$$K(\mathcal{O}_{F,S}) \longrightarrow \bigoplus_{\nu \in S} K(F_\nu) \xrightarrow{\bigoplus J_{F_\nu}} \mathrm{Pic}(\mathbb{S}_p).$$

Here $\mathcal{O}_{F,S} = \{x \in F \mid |x|_\nu \leq 1$ for all $\nu \notin S\}$.

Proof. Using that K-theory commutes with finite products, we can rewrite this as

$$K(\mathcal{O}_{F,S}) \longrightarrow K(\prod F_\nu) \xrightarrow{J} \mathrm{Pic}(\mathbb{S}_p).$$

Let us consider only the case of $\mathcal{O}_{F,S}$, general modules follow by tensoring. We need to show that the composition sends $\mathcal{O}_{F,S}$, which is given by $J(\prod F_\nu)$, to $\mathbb{S}_p$. Note that we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_{F,S} \longrightarrow \prod F_\nu \longrightarrow C_{F,S} \longrightarrow 0$$

where the first term is discrete, and the last is compact Hausdorff. By the previous theorem, all objects are vectorial, with the first etale and the last proper. By the theorem about proper/etale vectorial objects, the first and last are canonically sent $\mathbb{S}_p$, and since K-theory splits short exact sequences, we conclude the same for $\prod F_\nu$. ☐

3 Bonus: Class field theory and Artin reciprocity

In arXiv:1703.07842 (see also this talk), Dustin explains how to construct Artin map – the main construction of class field theory – using the ingredients of the second half of today. This is very cool, so I decided to include some discussion of that. Class field theory is in the business of understanding the maximal abelian Galois group G_F^{ab} of some fields, which turns out to be described by the very explicit idele class group C_F, which we divide to three cases:

$$C_F = \begin{cases} \mathbb{Z} & \text{finite field} \\ F^\times & \text{local field} \\ \mathbb{A}_F^\times / F^\times & \text{global field} \end{cases}$$

where the adeles are

$$\mathbb{A}_F := \prod{}'(F_\nu, \mathcal{O}_{F_\nu}) := \{(a_\nu) \in \prod F_\nu \mid a_\nu \in \mathcal{O}_{F_\nu} \text{ for all but finitely many } \nu\},$$

into which F embeds diagonally. For example, for $F = \mathbb{Q}$ this is $\mathbb{A}_\mathbb{Q} = (\widehat{\mathbb{Z}} \otimes \mathbb{Q}) \times \mathbb{R}$. The main theorem of class field theory is the following.

Theorem 3.1. *There is an Artin homomorphism*

$$C_F \longrightarrow G_F^{\mathrm{ab}},$$

inducing an isomorphism on the profinite completion of the source.

The Artin map is functorial in the field F in various ways, which also pins it down from the case of finite fields. A big part of the theorem is about producing the Artin map. We now explain a K-theoretic approach, which is uniform (and therefore functorial) in the field F. For simplicity, let's assume that the characteristic of F is not p (or throw away that part of the Galois group).

Proposition 3.2. *Assume that F has virtual cohomological dimension ≤ 2, then*

$$G_F^{\mathrm{ab}} \simeq \pi_1 \hom(L_{\mathrm{K}(1)}K(F), L_{K(1)} \operatorname{Pic}(\mathbb{S}_p)).$$

Proof idea. By Thomason's theorem, $L_{\mathrm{K}(1)}K$ satisfies etale hyperdescent. Thus, by Suslin's computation of K-theory of separably closed fields we can compute $L_{\mathrm{K}(1)}K(F)$ by descent. VCD condition guarantees the collapse of the spectral sequence giving

$$\pi_{-1}L_{\mathrm{K}(1)}K(F) = H^1(G_F, \mathbb{Z}_p) = \hom(G_F^{\mathrm{ab}}, \mathbb{Z}_p),$$

and the result follows by taking the dual. □

Proposition 3.3. *There is a canonical map $C_F \to \pi_1(K(\mathrm{LC}_F))$ where LC_F is locally compact condensed F-modules.*

Proof. This is where the product formula enters. Let me unpack this an argue more directly, and let's only do the most interesting one, the global case. Note $\mathbb{A}_F$ is an object of LC_F, with an action of $\mathbb{A}_F^\times$, giving $\mathbb{A}_F^\times \to \pi_1(K(\mathrm{LC}_F))$. It remains to show that it vanishes on $F^\times$. Note that we have an $F^\times$-equivariant short exact sequence

$$0 \longrightarrow F \longrightarrow \mathbb{A}_F \longrightarrow \mathbb{A}_F/F \longrightarrow 0.$$

But the first term is discrete and the last is compact, hence they vanish in K-theory, $F^\times$-equivariantly, and thus the same holds for $\mathbb{A}_F$, so $F^\times \to \mathbb{A}_F^\times \to \pi_1(K(\mathrm{LC}_F))$ is 0. ☐

Note that the tensor product of a locally compact condensed F-module with a finite dimensional F-vector space is locally compact, giving us

$$K(\mathrm{LC}_F) \otimes K(F) \longrightarrow K(\mathrm{LC}_F).$$

Post-compose with the J-homomorphism $K(\mathrm{LC}_F) \to \mathrm{Pic}(\mathbb{S}_p)$, and mate to get

$$K(\mathrm{LC}_F) \longrightarrow \hom(K(F), \mathrm{Pic}(\mathbb{S}_p)).$$

Since everything is functorial in F, which pins everything down, we get the following.

Corollary 3.4. *The Artin map is given by the composition*

$$C_F \to \pi_1(K(\mathrm{LC}_F)) \longrightarrow \pi_1 \hom(L_{\mathrm{K}(1)}K(F), L_{K(1)} \mathrm{Pic}(\mathbb{S}_p)) \simeq G_F^{\mathrm{ab}}.$$