# Constructing and (Not) Computing Algebraic K-Theory

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## 0 Zeroth K-group

### 0.1 Definition

If you walk around the mathematical universe, at some point you stumble upon the integers $\mathbb{Z}$. You may wonder how this got here. One way in which the integers arise is from the natural numbers $\mathbb{N}$. Indeed, recall that $(\mathbb{Z}, +)$ is formed from $(\mathbb{N}, +)$ by adjoining an inverse to every element. Putting this somewhat more abstractly, $(\mathbb{N}, +)$ from a commutative monoid, i.e. abelian group minus inverse axiom, and we can force it to be an abelian group, getting $(\mathbb{Z}, +)$.

$$\mathrm{CMon} \underset{\longleftarrow}{\overset{(-)^{\mathrm{gpc}}}{\underset{\perp}{\longrightarrow}}} \mathrm{Ab}$$

You might wonder, what other commutative monoids do I know? Well, before getting *other*, you can get $\mathbb{N}$ as follows. For a field $L$, finite dimensional vector spaces up to isomorphism are classified by their dimension, and the direct sum operation corresponds to to addition, i.e. $\mathbb{N} \simeq \mathrm{Vect}_L/\mathrm{iso}$. The group completion is then $\mathbb{Z} \simeq (\mathrm{Vect}_L/\mathrm{iso})^{\mathrm{gpc}}$.

One is then naturally led to consider the following. Let $R$ be a ring (unital, but not necessarily commutative). If we look at (finite dimensional) free modules $R^n$, we'll still get $\mathbb{N}$. Instead, we look at (finitely generated) projective modules

$$\mathrm{Proj}_R/\mathrm{iso} := \{M \in \mathrm{Mod}_R \mid \exists N \text{ such that } M \oplus N \simeq R^n\}.$$

*Remark* 0.1. As an algebro-geometric interlude, if $R$ is commutative, these are vector bundles over $\mathrm{Spec}(R)$. I do not wish to assume familiarity with this, but for some intuition, for any (closed) point, i.e. a maximal ideal $I \lhd R$, recall that the quotient $R/I$ is a field. Taking the quotient $M/IM$ then gives us a vector space over $R/I$. In other words, a projective $R$-module $M$ can be thought of as a collection of vector spaces $(V_I)$ that fit together.

**Definition 0.2.** We define the 0-th algebraic K group of $R$ to be the group-completion

$$\mathrm{K}_0(R) := (\mathrm{Proj}_R/\mathrm{iso}, \oplus)^{\mathrm{gpc}} \qquad \in \qquad \mathrm{Ab}.$$

**Example 0.3.** As we have seen above, for any field $L$, we have $\mathrm{K}_0(L) \simeq \mathbb{Z}$. More generally, for any principal ideal domain, such as $\mathbb{Z}$ or $L[x]$, projective implies free so

$$\mathrm{K}_0(R) \simeq \mathbb{N}^{\mathrm{gpc}} \simeq \mathbb{Z}.$$

One thing that I'd like to highlight about this construction is that it is in a sense transcendental: first since $R$ and $\mathrm{Proj}_R/\mathrm{iso}$ are of very different types and determining the latter from the former is very difficult; second since $(-)^{\mathrm{gpc}}$ is a very violent operation, since it adds both new generators and relations.

Another thing that I'd like to mention is that one of the main reasons to be interested in algebraic K-theory is because of its relation to other things. It is a natural home for many phenomena — let's see two examples.

## 0.2 Number theory: ideal class group

**Example 0.4.** Consider $R = \mathbb{Z}[\sqrt{-5}]$. It has a famous ideal $M = (2, 1+\sqrt{-5})$, which is a projective module which isn't free. But one can quite directly show that $M \oplus M \simeq R^2$. In fact, this is all there is:

$$\mathrm{K}_0(\mathbb{Z}[\sqrt{-5}]) \simeq \mathbb{Z} \oplus \mathbb{Z}/2.$$

This is a reflection of the fact that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain, as witnessed by $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.

Generalizing this example, given a number field $L$, consider the ring of integers $R = \mathscr{O}_L$, for which we have

$$\mathrm{K}_0(\mathscr{O}_L) \simeq \mathbb{Z} \oplus \mathrm{Cl}(L)$$

where $\mathrm{Cl}(L)$ is the ideal class group from number theory (more generally, for any Dedekind domain we get the Picard group $\mathrm{Pic}(R)$ from algebraic geometry). As suggested above, $\mathscr{O}_L$ is a UFD if and only if the group $\mathrm{Cl}(L)$ is trivial. Determining $\mathrm{Cl}(L)$, or even its size, is doable in concrete cases, but many basic questions are still open, for instance:

**Conjecture 0.5** (Class number problem)**.** *Are there infinitely many $d \in \mathbb{N}$ such that $\mathrm{Cl}(\mathbb{Q}(\sqrt{d}))$ is trivial?*

Note that we always have the free boring modules $R^n$, which forms a subgroup $\mathbb{Z} \leq \mathrm{K}_0(R)$. We denote by $\widetilde{\mathrm{K}}_0(R)$ the quotient by them, so that for example in the last example $\widetilde{\mathrm{K}}_0(R) \simeq \mathrm{Cl}(R)$.

## 0.3 Topology: Wall's finiteness obstruction

Without being too precise, recall that a finite CW complex is a topological space made by gluing finitely many disks together. Examples include: $n$-disk, $n$-sphere, the shapes $\times, \perp$, $n$-projective space, graphs, $n$-torus, smooth closed manifolds, and similar "reasonable" topological spaces.

**Definition 0.6.** A space $X$ is called finitely dominated if it is a retract of a CW complex up to homotopy, that is, if there is a CW complex $Y$ and $i\colon X \to Y, r\colon Y \to X$ such that $X \xrightarrow{i} Y \xrightarrow{r} X$ is homotopic to $\mathrm{Id}_X$.

**Question 0.7.** *Is any finitely dominated $X$ (homotopy equivalent to) a CW complex?*

This question came up in Wall's work on surgery theory. He was trying to construct closed manifolds, but was able to construct finitely dominated spaces, and wondered what is the difference. In fact, he found a characterization. Look at the fundamental group (at some arbitrary base-point, and assume $X$ is connected)

$$\pi_1(X, x) = \{\gamma\colon [0,1] \longrightarrow X \mid \gamma(0), \gamma(1) = x\}/\text{homotopy},$$

which I remind you is the group of loops with respect to concatenation. Consider the ring

$$\mathbb{Z}[\pi_1(X)] := \{\sum a_i[\gamma_i] \mid a_i \in \mathbb{Z}, [\gamma_i] \in \pi_1(X)\}.$$

The answer lies in the K-theory of this ring. Wall has explicitly constructed an element $o(X) \in \mathrm{K}_0(\mathbb{Z}[\pi_1(X)])$ (this invariant is very related to the Euler characteristic of $X$, indeed the map $\mathrm{K}_0(\mathbb{Z}[\pi_1(X)]) \to \mathrm{K}_0(\mathbb{Z}) \simeq \mathbb{Z}$ sends $o(X)$ to $\chi(X)$).

**Theorem 0.8** (Wall)**.** *A finitely dominated space $X$ is equivalent to a CW complex if and only if the image $\widetilde{o}(X) \in \widetilde{\mathrm{K}}_0(\mathbb{Z}[\pi_1(X)])$ vanishes.*

**Corollary 0.9.** *If $X$ is a simply connected finitely dominated space, then it is equivalent to a CW complex.*

*Proof.* Indeed, simply connected means $\pi_1(X) = 1$, so that $\widetilde{\mathrm{K}}_0(\mathbb{Z}[\pi_1(X)]) = \widetilde{\mathrm{K}}_0(\mathbb{Z})$ which is a trivial group, and in particular $\widetilde{o}(X)$ vanishes. ☐

# 1 First K-group

## 1.1 Definition

Some time in the past, one of my favorite professors from Hebrew University told me that he has been saying for years:

> The point of undergraduate studies is that isomorphic objects are the same.
> The point of graduate studies is that isomorphic objects are not the same.

We shall follow this mantra. As the notation suggests, $\mathrm{K}_0(R)$ is only the zeroth among an infinite list of interesting groups $\mathrm{K}_n(R)$ associate to $R$, let us move on to understand $\mathrm{K}_1(R)$. Recall that we defined

$$\mathrm{K}_0(R) := (\mathrm{Proj}_R/\mathrm{iso}, \oplus)^{\mathrm{gpc}}.$$

Observe that exactly neglected all the isomorphisms! Let me say that for the purposes of the higher K-groups you may consider only the "boring" free modules $R^n$. While their structure modulo isomorphism is very simple (just $\mathbb{N}$, giving $K_0 = \mathbb{Z}$), there are many interesting isomorphisms: indeed, the automorphisms of $R^n$ is the group of invertible $n \times n$-matrices $\mathrm{GL}_n(R)$, which is an extremely interesting and rich object. Thus, we wish to retain all of this information. This is achieved by considering the groupoid $\mathrm{Proj}_R$. Let's briefly recall that a groupoid is a category with all morphisms invertible, that is

(1) a collection of objects $\{X\}$,

(2) morphisms between them $\{X \xrightarrow{f} Y\}$ and special ones $X \xrightarrow{\mathrm{Id}_X} X$,

(3) which we can compose $\{X \xrightarrow{f} Y \xrightarrow{g} Z\}$,

(4) such that for any $X \xrightarrow{f} Y$ there is $Y \xrightarrow{g} X$ such that $gf = \mathrm{Id}_X$ and $fg = \mathrm{Id}_Y$.

I hope that it is clear that $\mathrm{Proj}_R$ forms a groupoid, where the objects are the projective $R$-modules $M$, and an morphisms is an isomorphism $M \xrightarrow{f} N$ (and isomorphism exactly means that there is an inverse).

Now, recall that the important bit for $K_0$ was the direct sum. Importantly for us, we can also direct sum homomorphisms: indeed, given $M \xrightarrow{f} N$ and $M' \xrightarrow{f'} N'$ we have

$$M \oplus M' \xrightarrow{f \oplus f'} N \oplus N', \qquad (f \oplus f')(m, m') = (f(m), f'(m')).$$

**Example 1.1.** Considering free modules, for $A \in \mathrm{GL}_n(R)$ and $B \in \mathrm{GL}_k(R)$ we can form the block matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{GL}_{n+k}(R)$.

Summing up, $\mathrm{Proj}_R \in \mathrm{CMon(Grpd)}$ is a commutative monoid (i.e. we can sum, but not subtract) which is a groupoid (i.e. we remember the isomorphisms between objects). Just like before, we can group-complete it, forcefully adjoining an inverse to every object

$$\mathrm{CMon(Grpd)} \underset{\longleftarrow}{\overset{(-)^{\mathrm{gpc}}}{\underset{\perp}{\longrightarrow}}} \mathrm{Ab(Grpd)}$$

**Definition 1.2.** $K^{\leq 1}(R) := (\mathrm{Proj}_R, \oplus)^{\mathrm{gpc}} \qquad \in \qquad \mathrm{Ab(Grpd)}$.

*Remark* 1.3. This object/notation is very non-standard, and we will only temporarily consider it.

Well, what is this? First, note that since it is a groupoid, we can look at objects up to isomorphism, and it is immediate from the definition that

$$K_0(R) \simeq K^{\leq 1}(R)/\mathrm{iso}.$$

So if we kill the isomorphisms, there is nothing new, which is good. What about the automorphisms of objects in this groupoid? It turns out that it doesn't matter which object you choose, they all have the same automorphisms, and they are all abelian groups (both follow from Eckmann–Hilton).

**Definition 1.4.** We define $\mathrm{K}_1(R) \in \mathrm{Ab}$ to be the automorphisms of any object in $\mathrm{K}^{\leq 1}(R)$.

So the discussion above was quite abstract, begging the question, what is this group? As I mentioned above, it is related to $\mathrm{GL}_n(R)$, but somehow for all $n$ simultaneously. Indeed, consider the group $\mathrm{GL}(R) := \bigcup \mathrm{GL}_n(R)$ of infinite matrices (but non-zero only in finitely many entries, i.e., bounded matrices). The trouble is that this group is not abelian, while $\mathrm{K}_1(R)$ is.

**Proposition 1.5.** *The first K-group is the abelianization*

$$\mathrm{K}_1(R) \simeq \mathrm{GL}(R)^{\mathrm{ab}} = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)].$$

In general, this group may be very very complicated. Let us see however another connection to number theory.

## 1.2 Number theory: analytic class number formula

Note however that we have $R^\times = \mathrm{GL}_1(R) \leq \mathrm{GL}(R)$, and if $R$ is commutative ring, this is an abelian group. One might hope that this happens to be $\mathrm{K}_1(R)$, and this indeed happens sometimes.

**Theorem 1.6** (Bass–Milnor–Serre)**.** *For a number field $L$*

$$\mathrm{K}_1(\mathscr{O}_L) \simeq \mathscr{O}_L^\times.$$

**Example 1.7.** For $L = \mathbb{Q}$, we get

$$\mathrm{K}_1(\mathbb{Z}) \simeq \mathbb{Z}^\times = \{\pm 1\} \simeq \mathbb{Z}/2.$$

The classical Dirichlet's unit theorem describes $\mathscr{O}_L^\times$. It is a finitely generated abelian group, so it is of the form $\mathbb{Z}^r \oplus$ torsion. The torsion elements, i.e. elements such that $x^k = 1$ for some $k$, are by definition the roots of unity in $\mathscr{O}_L$. The number $r$ can also be described (via the real and complex embeddings of $\mathscr{O}_L$).

In fact, this allows us to relate K-groups to zeta functions. Recall Riemann's zeta function can be expressed as

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}.$$

This is in fact the zeta function associated to $\mathbb{Q}$, indeed noting the appearance of the (ordinary) primes. By analogy, one studies the zeta function associated to any number field $L$

$$\zeta_L(s) = \prod_{\mathfrak{p} \lhd \mathscr{O}_L} \frac{1}{1 - |\mathscr{O}_L/\mathfrak{p}|^{-s}}.$$

As is the case for Riemann's zeta function, this function converges for all Real$(s) > 1$ and admits a meromorphic continuation, with one pole, at 1. This pole is simple, i.e. the function around it is

$$a_{-1}\frac{1}{(z-1)} + a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots .$$

The first coefficient $\mathrm{sv}_1(\zeta_L) := a_{-1}$, which is called the special value at $s = 1$, which a priori is mysterious, was computed by Dirichlet's analytic class number formula in terms interesting invariants attached to $L$. Given our descriptions above, this can be restated as follows.

**Theorem 1.8.** *The special value of* $\zeta_L$ *at* $s = 1$ *is given by*

$$\mathrm{sv}_1(\zeta_L) = -\frac{|\mathrm{K}_0(\mathscr{O}_L)_{\mathrm{tors}}|}{|\mathrm{K}_1(\mathscr{O}_L)_{\mathrm{tors}}|}R_1(L)$$

*where the subscript* tors *denotes the torsion subgroup, and* $R_1(L)$ *is the regulator of* $L$.

# 2 Higher K-groups and higher category theory

Recall that to define the zeroth and first K-groups, we took the groupoid $\mathrm{Proj}_R$, that is the projective $R$-modules together with their isomorphisms, together with the direct sum operation, and group-completed it. $\mathrm{K}_0(R)$ is the isomorphism classes, and $\mathrm{K}_1(R)$ are the automorphisms of (any) object. Where can we expect to find more K-groups? What extra input we didn't use? The answer is that no new input is needed! Instead, we shall make use of the tools of homotopy theory, specifically, $\infty$-groupoids.

## 2.1 $\infty$-groupoids

In the interest of time, I will only give one motivation/example. Given a topological space $X$ with a base-point $x$, recall that we have the fundamental group $\pi_1(X, x)$; I'd like to highlight two things:

(1) We had to choose $x \in X$, and we look at path starting and ending at $x$.

(2) Paths are up to homotopy.

I'd like to eliminate these two points. Let's start with the first: choosing a base-point is somewhat artificial, and in fact, this is quite easy to remedy. We define the fundamental *groupoid* $\Pi_1(X)$ to have

- Objects: points $x \in X$.
- Morphisms: paths $\gamma\colon x \rightsquigarrow y$ up to homotopy, whose composition is given by concatenation.

Observe that if we take any $x \in X$, then the automorphisms of $x$ in $\Pi_1(X)$ are precisely $\pi_1(X, x)$, so this is indeed a fix of the first point.

Moving on to the second point. We'd like to get rid of the "up to homotopy". Recall that a homotopy $H$ from $\gamma$ to $\gamma'$ is, in fact, nothing more than a path from $H \colon \gamma \rightsquigarrow \gamma'$. Why don't we just incorporate these? Indeed, we can define the fundamental 2-groupoid $\Pi_2(X)$, which has one extra layer of morphisms.

- Objects: points $x \in X$.
- Morphisms: paths $\gamma \colon x \rightsquigarrow y$, whose composition is given by concatenation.
- 2-morphisms: homotopies $H \colon \gamma \rightsquigarrow \gamma'$, whose composition is given by concatenation.

I hope that the pattern is clear, we define the $\infty$-groupoid $\Pi_\infty(X)$, which has $n$-morphisms, where the next step is

- 3-morphisms: homotopies $K \colon H \rightsquigarrow H'$, whose composition is given by concatenation.
- ...

Of course, I didn't tell you what precisely is an $\infty$-groupoid, but it is some mathematical object which has objects, 1-morphisms, 2-morphisms, ..., which can be composed an so on. $\Pi_\infty(X)$ is the prototypical example of $\infty$-groupoid. In fact, this is a theorem.

**Theorem 2.1.** *Any $\infty$-groupoid is isomorphic to $\Pi_\infty(X)$ for some topological space $X$.*

So you actually "know" all of them.

## 2.2 Definition of K-groups

Going back to algebraic K-theory, as I mentioned, we are not going to have any new input. Recall that we have $(\mathrm{Proj}_R, \oplus) \in \mathrm{CMon}(\mathrm{Grpd})$. Previously we have group-completed it. Note that just like any set is a groupoid (where the only morphisms are the identities), any groupoid is an $\infty$-groupoid (where the only higher morphisms are the identities). Thus, we can consider $(\mathrm{Proj}_R, \oplus)$ as an $\infty$-groupoid with a summation operation. Once more, we can group-complete

$$\mathrm{CMon}(\mathrm{Grpd}_\infty) \underset{\longleftarrow}{\overset{(-)^{\mathrm{gpc}}}{\underset{\perp}{\longrightarrow}}} \mathrm{Ab}(\mathrm{Grpd}_\infty)$$

**Definition 2.2.** $\mathrm{K}(R) := (\mathrm{Proj}_R, \oplus)^{\mathrm{gpc}} \quad \in \quad \mathrm{Ab}(\mathrm{Grpd}_\infty).$

From this we can extract the higher groups as the higher automorphisms of any arbitrary $n$-morphism (as before, this doesn't depend on which one you choose). In fact, as we said above any $\infty$-groupoid comes from a topological space, and these will be just be its homotopy groups.

**Definition 2.3.** $\mathrm{K}_n(R) := \pi_n(\mathrm{K}(R)) \quad \in \quad \mathrm{Ab}.$

It is perhaps surprising that we'd get something new like this — it is natural to expect that since we started with a normal groupoid we'd get a normal groupoid, but this is not the case. The reason is that the operation $(-)^{\mathrm{gpc}}$ is very violent (to those of you to whom this means something, this is similar to a left derived functor, and indeed there are infinitely many derived functors). Perhaps it is then not surprising that these K-groups are extremely difficult to compute.

## 2.3 Finite fields

There is in fact only one fundamental computation in the subject which we understand completely, which is an amazing theorem of Quillen. We already know $\mathrm{K}_0(\mathbb{F}_q) \simeq \mathbb{Z}$ and $\mathrm{K}_1(\mathbb{F}_q) \simeq \mathbb{F}_q^\times \simeq \mathbb{Z}/q-1$.

**Theorem 2.4** (Quillen)**.** $\mathrm{K}_{2i}(\mathbb{F}_q) = 0$ *and* $\mathrm{K}_{2i-1}(\mathbb{F}_q) = \mathbb{Z}/q^i - 1$.

The proof really is quite amazing, using tools from topology, higher category theory and modular representation theory. In fact, the proof proceeds by a comparison to $\mathrm{K}(\mathbb{C})$!

## 2.4 The integers

To illustrate just how difficult it is to compute K-theory, here are the first $\mathrm{K}_n(\mathbb{Z})$.

$$\mathrm{K}_0(\mathbb{Z}) = \mathbb{Z}, \qquad \mathrm{K}_1(\mathbb{Z}) = \mathbb{Z}/2, \qquad \mathrm{K}_2(\mathbb{Z}) = \mathbb{Z}/2, \qquad \mathrm{K}_3(\mathbb{Z}) = \mathbb{Z}/48, \qquad \mathrm{K}_4(\mathbb{Z}) = 0.$$

By now all K-groups of $\mathbb{Z}$ are known, and are related to the Bernoulli numbers, except for $\mathrm{K}_{4n}(\mathbb{Z})$. The Vandiver conjecture from number theory is equivalent to $\mathrm{K}_{4n}(\mathbb{Z}) = 0$, and the second case $\mathrm{K}_8(\mathbb{Z}) = 0$ was only proven in a breakthrough work in 2018.

## 2.5 More zeta functions

We can consider again the ring of integers of a number field $\mathscr{O}_L$. Recall that we previously mentioned that $\zeta_L$ has a pole of order 1 at $s = 1$, and leading coefficient of the Taylor series is

$$\mathrm{sv}_1(\zeta_L) = -\frac{|\mathrm{K}_0(\mathscr{O}_L)_{\mathrm{tors}}|}{|\mathrm{K}_1(\mathscr{O}_L)_{\mathrm{tors}}|} R_1(L).$$

These two phenomena have a conjectural extension to the zeros at negative integers.

**Conjecture 2.5** (Soulé, Lichtenbaum)**.** *The zeta function $\zeta_L$ vanishes at $-n$ to order $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes \mathrm{K}_{2n+1}(\mathscr{O}_L))$. And the leading coefficient of the Taylor series is*

$$\mathrm{sv}_{-n}(\zeta_L) = \pm \frac{|\mathrm{K}_{2n}(\mathscr{O}_L)_{\mathrm{tors}}|}{|\mathrm{K}_{2n+1}(\mathscr{O}_L)_{\mathrm{tors}}|} R_{n+1}(L).$$

# 3 Epilogue

As I've mentioned several times, computing K-groups is extremely difficult. I'd like to very briefly comment on two approaches to doing so.

## 3.1 Descent

One common strategy to solve problems is a divide and conquer approach. In (algebraic) geometry this manifest itself in descent. Indeed, recall that to a commutative ring $R$ there is an associated space $\mathrm{Spec}(R)$. One can try to cover $\mathrm{Spec}(R)$ by (hopefully) simpler spaces, compute their K-theory, and glue the results. This indeed works, technically referred to as Zariski descent (K-theory even satisfies Nisnevich descent).

However, perhaps the simplest way to simplify a field $L$ is by passing to a Galois extension or to its algebraically closure altogether $\overline{L}$ (this indeed fits the picture in algebraic geometry, where this is pictured as a covering space with fibers $\mathrm{Gal}(\overline{L}/L)$). Algebraically closed fields are indeed way simpler, for instance, in linear algebra, any matrix can be put into Jordan form. Indeed, $\mathrm{K}(\overline{L})$ was almost completely computed by Suslin, and it almost doesn't depend on the specific field!

One would then hope that we can recover $\mathrm{K}(L)$ from $\mathrm{K}(\overline{L})$ together with its action by $\mathrm{Gal}(L'/L)$. This, (un)fortunately fails: it turns out that K-theory does *not* satisfy Galois descent, which is a very interesting and subtle phenomenon. Indeed, K-theory *almost* satisfies Galois descent. This is codified by the Quillen–Lichtenbaum conjecture, proven by Voevodsky, which is a beautiful story relating algebraic K-theory to etale cohomology.

It was also explained by Waldhausen (building on the work of Thomason) how to give a homotopical description of the Quillen–Lichtenbaum conjecture in chromatic homotopy theory. This was later vastly generalized by the influential redshift conjecture of Ausoni–Rognes, which is one my own personal interests.

## 3.2 Trace methods

The other key approach to study K-theory is via what's called trace methods. These are approximations of K-theory, analogues to approximating a linear transformation by its trace. This has applications too many to name. To illustrate, a very recent and easy to state result of Antieau–Krause–Nikolaus from 2024 gives an algorithm for computing $\mathrm{K}_i(\mathbb{Z}/p^n)$, as well as some structural results, via approximating K-theory by prismatic and syntomic cohomology.