Redshift, blueshift, and traces

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This work is heavily influenced by, and partly joint with Lior Yanovski.

1 Introduction

In this talk I would like to explain some work in progress, and along the way survey various exciting results in chromatic homotopy theory and algebraic K-theory from recent years.

Goal 1. Relate the transchromatic and Dennis trace maps

$$\chi\colon \mathrm{E}_n^X \longrightarrow C_t^{L_p^{n-t}X}, \qquad \mathrm{K}(\mathrm{Mod}_R^X) \longrightarrow \mathrm{THH}(\mathrm{Mod}_R^X).$$

I'll start with a few words on the space X. Being a bit philosophical, spaces can be thought of as built in either of the following ways:

(1) Cellular structure; building blocks are spheres S^n; finite = finitely many cells.

(2) Postnikov tower; building blocks are $\mathrm{B}^n G$; finite = π-finite, i.e., the entire $\pi_* X$ is finite.

Many constructions in stable homotopy theory are exact, that is, they preserve finite limits and colimits. Loosely interpreted, this means that they behave in a controlled way with respect to the first perspective on spaces. My focus is thus on the latter, so for this talk X is a π-finite space, and the example to have in mind is $X = \mathrm{B}G$ for a finite group.

Taking this hint, let us start with the common origin of both maps above. One of the main tools to study the representations of a finite group G is the character map

$$\chi\colon \mathrm{Vect}_{\mathbb{C}}^{\mathrm{B}G,\mathrm{gpc}} \longrightarrow \mathbb{C}^{G/\mathrm{conj}}, \qquad \chi_V([g]) \;=\; \mathrm{tr}(g\colon V \to V).$$

For example, its mate under the free-forgetful adjunction is an isomorphism

$$\mathbb{C} \otimes \mathrm{Vect}_{\mathbb{C}}^{\mathrm{B}G,\mathrm{gpc}} \xrightarrow{\sim} \mathbb{C}^{G/\mathrm{conj}}.$$

The appearance of G/conj has a homotopical explanation. Indeed, the free loop space is

$$L\mathrm{B}G \simeq \hom(\mathrm{B}\mathbb{Z}, \mathrm{B}G) \simeq \hom_*(\mathrm{B}\mathbb{Z}, \mathrm{B}G)/\!/\mathrm{conj} \simeq \hom_*(\mathbb{Z}, G)/\!/\mathrm{conj} \simeq G/\!/\mathrm{conj}.$$

Since the rational cohomology of any finite group vanishes, $\mathbb{C}$ doesn't see the difference between $L\mathrm{B}G \simeq G/\!/\mathrm{conj}$ and G/conj. This allows us to write the target of character map as $\mathbb{C}^{L\mathrm{B}G}$, which is more amenable to generalization.

2 Transchromatic character map

Somewhat loosely speaking, chromatic homotopy theory provides the height filtration $L_n\mathrm{Sp}$ on p-local spectra (fixing an implicit prime p), whose associated graded are the $\mathrm{K}(n)$-local spectra $\mathrm{Sp}_{\mathrm{K}(n)}$.

Example 2. Height $n = 0$ is rational spectra $\mathrm{Sp}_{\mathrm{K}(0)} = \mathrm{Sp}_{\mathbb{Q}} = \mathcal{D}(\mathbb{Q})$.

As in ordinary algebra, it is often useful to pass to algebraic closure. In this case, the algebraic closure of the unit $\mathbb{S}_{\mathrm{K}(n)}$ is the Lubin–Tate (or Morava E-theory) spectrum E_n.

Example 3. At height $n = 1$ this is complex K-theory $\mathrm{E}_1 = \mathrm{KU}_p$ (up to adding roots of unity, if we really want algebraic closure).

We are thus naturally led to study the cohomology E_n^X. Let us start with height 1. Recall that there is a relation between representations and vector bundles via $G \to \mathrm{U}(n) \iff \mathrm{B}G \to \mathrm{BU}(n)$. Indeed, the Atiyah–Segal theorem says that $\mathrm{KU}^{\mathrm{B}G}$ is the representation ring $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{B}G,\mathrm{gpc}}$, up to completion and 2-periodization. We have observed that $\mathrm{E}_1 = \mathrm{KU}_p$, and we can reinterpret the character map as

$$\chi\colon \mathrm{E}_1^{\mathrm{B}G} \longrightarrow \mathrm{P}\mathbb{Q}_p(\zeta_{p^\infty})^{L_p\mathrm{B}G}.$$

Note that we modified the target. Since we are working p-completely, we only see p-power-torsion elements of G, hence the p-adic free loop space $L_p\mathrm{B}G := \hom(\mathrm{B}\mathbb{Z}_p, \mathrm{B}G)$. Recall that in character theory, the eigenvalues of any representations are in the cyclotomic extension so we don't really need all of $\mathbb{C}$, and since we are in the p-typical setting p^∞-roots of unity suffice. Since $\mathrm{E}_1 = \mathrm{KU}_p$ is p-complete 2-periodic, we get 2-periodic $\mathbb{Q}_p(\zeta_{p^\infty})$. To sum, character theory allows us to understand height 1 using height 0.

The generalization is then natural: we would like to have a higher height form of character theory, that allows us to study height n using lower heights $t \leq n$ information. We thus replace E_1 by E_n, and similarly replace $\mathrm{P}\mathbb{Q}_p = L_{\mathrm{K}(0)}\mathrm{E}_1$ by $L_{\mathrm{K}(t)}\mathrm{E}_n$. Observe that $\mathrm{P}\mathbb{Q}_p(\zeta_{p^\infty})$ is obtained by adjoining roots of unity, that is, torsion points of the multiplicative formal group, the Quillen formal group of KU_p. Similarly C_t is an $L_{\mathrm{K}(t)}\mathrm{E}_n$-algebra obtained by adjoining torsion points of the Quillen formal group of E_n. Finally, this works for any π-finite space X, such as $\mathrm{B}G$. The transchromatic character map takes the form

$$\chi\colon \mathrm{E}_n^X \longrightarrow C_t^{L_p^{n-t}X}.$$

While its construction is quite simple, the main theorem of Hopkins–Kuhn–Ravenel, Stapleton and Lurie is the analogue of what we had for the character map.

Theorem 4. *The mate under the free-forgetful adjunction is an isomorphism*

$$C_t \otimes_{\mathrm{E}_n} \mathrm{E}_n^X \xrightarrow{\sim} C_t^{L_p^{n-t}X}.$$

3 Dennis trace map

The character map has a generalization in a different direction. Recall that for a stable category $\mathcal{C}$, we have the Dennis trace map

$$\mathrm{K}(\mathcal{C}) \longrightarrow \mathrm{THH}(\mathcal{C}).$$

This is a refinement and a generalization of the case $\mathcal{C} = \mathrm{Perf}(R)$ for a commutative R, where the map $\mathrm{K}(R) \to \mathrm{THH}(R)$ factors the dimension map

$$\dim\colon \mathrm{Perf}(R)^{\simeq} \longrightarrow R, \qquad \dim(M) \ := \ (R \to M \otimes M^{\vee} \simeq M^{\vee} \otimes M \to R).$$

Famously, taking $R = \mathbb{C}[G]$, this recovers the character map on π_0. On the K-side this is almost by definition, and on the HH it uses the well-known result

$$\mathrm{THH}(R[G]) \simeq \mathrm{THH}(R)[LBG],$$

or the direct computation

$$\mathrm{THH}_0(\mathbb{C}[G]) \simeq \mathbb{C}[G]/[\mathbb{C}[G], \mathbb{C}[G]] \simeq \mathbb{C}[G/\mathrm{conj}].$$

Observation 5. Canonically, we get the colimit rather than the limit over G/conj. Since this is a finite set, semiadditivity gives a canonical isomorphism between them. Relatedly, note that the character map is a commutative ring map, whereas in this treatment it is unclear how to retain the multiplicative structure. We will return to this point later.

4 The sources, redshift and blueshift

In the course of this section, I would like to relate the sources of the two trace maps, and in the section turn to the targets. Before giving more details on this, let me zoom out and go back to our goal. The Ausoni–Rognes redshift conjecture, now a theorem for $\mathbb{E}_\infty$-ring due to Clausen–Mathew–Naumann–Noel, Land–Mathew–Meier–Tamme, Yuan and Burklund–Schlank–Yuan.

Theorem 6. *Algebraic K-theory increases chromatic height by* 1*, while* THH *does not.*

Example 7. Up to completion $\mathrm{K}(\mathbb{C}) \simeq \mathrm{KU}$ (height 1), while $\mathrm{THH}(\mathbb{C})$ is rational (height 0).

Consequently, the Dennis trace map $\mathrm{K} \to \mathrm{THH}$ is comparing height $n+1$ and height n. Thus, our goal is to relate it to the transchromatic character for these heights.

Goal 8. Relate the transchromatic and Dennis trace maps

$$\chi\colon \mathrm{E}_{n+1}^X \longrightarrow C_n^{L_p X}, \qquad \mathrm{K}(\mathrm{Mod}_R^X) \longrightarrow \mathrm{THH}(\mathrm{Mod}_R^X).$$

Let us turn out attention to relating the sources of these maps, starting with the simplifying assumption $X = \mathrm{pt}$, so we would like to relate E_{n+1} to some $\mathrm{K}(R)$. To that end, we will use the Tate construction of a spectrum M with a G-action: the quotient of M^{hG} by elements induced from the trivial subgroup (in the same sense of induced representations), that is, the cofiber

$$M^{tG} := \mathrm{cofib}(M_{hG} \xrightarrow{\mathrm{Nm}} M^{hG}).$$

It is very common to take this respect to the trivial G-action, in which case we have the following result of Greenlees–Hovey–Sadofsky and Kuhn, known as Tate blueshift.

Theorem 9. *Tate construction with respect to a finite group decreases chromatic height by* 1.

Example 10. $\mathrm{KU}_p^{t\mathbb{Z}/p} \simeq \mathrm{P}\mathbb{Q}_p(\zeta_p)$.

Crucially however, Tate construction does *not* decrease for S^1.

Example 11. If E is complex oriented, then $\pi_*(E^{tS^1}) \simeq E_*((x))$. Thus, they are roughly the same (for example E is a retract of E^{tS^1}), and indeed have the same height.

With this in mind, Yuan proves the following example of redshift.

Theorem 12. *Let E be an $\mathbb{E}_\infty$-ring of height $n+1$. Then $\mathrm{K}(E^{t\mathbb{Z}/p})$ is also of height $n+1$.*

Proof sketch. The bound $\leq n+1$ is due to CMNN+LMMT, so we prove $\geq n+1$. Recall that THH admits an S^1-action, for which the Dennis trace map is invariant. Now, for an $\mathbb{E}_\infty$-ring we have $\mathrm{THH}(A) \simeq S^1 \otimes A$, i.e. it is free $\mathbb{E}_\infty$-ring with an S^1-action on A. Consequently, if A itself is endowed with an S^1-action, then we have the counit of the free–forgetful adjunction, giving an S^1-equivariant map $\mathrm{THH}(\underline{A}) \to A$, hence a map $\mathrm{THH}(\underline{A})^{hS^1} \to A^{hS^1}$.

Endow E with the trivial S^1-action, and let $A := E^{t\mathbb{Z}/p}$ with the residual action by $S^1/(\mathbb{Z}/p) \simeq S^1$. We thus get

$$\mathrm{K}(\underline{E^{t\mathbb{Z}/p}}) \longrightarrow \mathrm{THH}(\underline{E^{t\mathbb{Z}/p}})^{hS^1} \longrightarrow (E^{t\mathbb{Z}/p})^{hS^1} \approx E^{tS^1}.$$

For the last step, we use the Nikolaus–Scholze Tate-orbit lemma which says that the two agree up to p-completion. This requires E to be connective, which we may assume since the LMMT purity theorem says that this won't change $\mathrm{K}(n+1)$-local K-theory.

Finally, as noted above, E and E^{tS^1} have the same height, so that E^{tS^1} does not vanish $\mathrm{K}(n+1)$-locally. This implies that the source $\mathrm{K}(\underline{E^{t\mathbb{Z}/p}})$ also doesn't vanish $\mathrm{K}(n+1)$-locally, since these are ring maps and there are no maps from the zero ring to a non-zero ring. □

Loosely speaking, this proof shows that $\mathrm{K}(\mathrm{E}_{n+1}^{t\mathbb{Z}/p})$ (which also works verbatim for $\mathrm{E}_{n+1}^{t\mathbb{Z}/p^r}$) can be compared to E_{n+1}, or more precisely to $\mathrm{E}_{n+1}^{tS^1}$.

Let us now incorporate the space X. To that end, note that we are trying to map to something of height $n+1$, so we may consider $\mathrm{K}(n+1)$-local K-theory. Then, the higher descent theorem by myself, Carmeli, Schlank and Yanovski, generalizing a theorem by CMNN, says:

Theorem 13. *Let $\mathcal{C}$ be a category height $\leq n$ (such as $\mathrm{Perf}(R)$ for R of height $\leq n$), then*

$$L_{\mathrm{K}(n+1)}\mathrm{K}(\mathcal{C}^X) \xrightarrow{\sim} L_{\mathrm{K}(n+1)}\mathrm{K}(\mathcal{C})^X.$$

5 The targets

The discussion above suggests taking $R = \mathrm{E}_{n+1}^{t\mathbb{Z}/p^r}$ (with $r \to \infty$). I have not described the construction of C_t, but it is in fact obtained from this by further manipulations. This ties the targets of the two trace maps for $X = \mathrm{pt}$.

Let us now try to incorporate X. As I have alluded to before, there is a well-known result of Goodwillie–Jones, saying that for any $\mathcal{C} \in \mathrm{Cat}_{\mathrm{st}}$ and space X we have

$$\mathrm{THH}(\mathcal{C}[X]) \simeq \mathrm{THH}(\mathcal{C})[LX].$$

Problem 14. Both sides are colimits rather than limits.

In the height 0 case we used semiadditivity $\mathbb{C}[Y] \simeq \mathbb{C}^Y$. Similarly, working at height n, we have the following result proven by Hopkins–Lurie, then reproved and extended by Carmeli–Schlank–Yanovski, and reproved again by myself, which allows us to immediately deal with the right hand side after $\mathrm{K}(n)$-localization (and is also essential for the higher descent theorem).

Theorem 15. *The category $\mathrm{Sp}_{\mathrm{K}(n)}$ is ∞-semiadditive. That is, for any local system $M \colon X \to \mathrm{Sp}_{\mathrm{K}(n)}$ indexed on a π-finite space X, there is a canonical isomorphism*

$$\operatorname*{colim}_X M \xrightarrow{\sim} \lim_X M.$$

I would like to explain how to handle the left hand side. This is most easily understood by passing to the large categories via $\mathcal{D} := \mathrm{Ind}(\mathcal{C})$. Indeed, recall that it induces an equivalence $\mathrm{Ind} \colon \mathrm{Cat}_{\mathrm{perf}} \xrightarrow{\sim} \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$(where the latter denotes compactly generated categories and left adjoint functors preserving compact objects).In this language THH becomes the dimension

$$\mathrm{THH}(\mathcal{C}) \simeq \dim(\mathcal{D}) \qquad \in \qquad \mathrm{End}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}}(\mathrm{Sp}) \simeq \mathrm{Sp}.$$

Famously, Lurie proved that limits and colimits in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$, indexed on any space, are the same, so $\mathcal{D}[X] \simeq \mathcal{D}^X$ as computed in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$. This is however *not* true in the (non-full non-wide) subcategory $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$. Nevertheless, I have proven together with Carmeli, Schlank and Yanovski that this does hold for $\mathrm{K}(n)$-linear categories and π-finite spaces.

Theorem 16. *The category $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{K}(n),\omega}$ is ∞-semiadditive.*

This essentially explains how to handle the left hand side as well. However something is left to be desired: the functoriality in X and the multiplicative structures in either side, are somewhat unclear. To that end, combining the last result, together with the universal property of the 2-category of iterated spans, conjectured by myself and proven by Cnossen–Lenz–Linskens, we can prove the following tighter result.

Theorem 17. *There is an isomorphism*

$$\mathrm{HH}_{\mathrm{K}(n)}(\mathcal{D}^X) \simeq \mathrm{HH}_{\mathrm{K}(n)}(\mathcal{D})^{LX}$$

symmetric monoidally natural in $\mathcal{D} \in \mathrm{Pr}^{\mathrm{L}}_{\mathrm{K}(n),\omega}$ and $X \in \mathrm{Span}(\mathcal{S}_{\pi\text{-fin}})$.

6 Summary and loose ends

Let me conclude by tying these threads back to our goal – note though that this is still in progress, so the full picture is not yet clear (though clearer than I had time to portray).

On the source side, Yuan's argument compared the K-theory of $\mathrm{E}_{n+1}^{t\mathbb{Z}/p^r}$ to E_{n+1} (albeit after a mild S^1-Tate construction). The higher descent theorem then lets us incorporate the space X. On the target side, the last result deals with the dependence on X.

One thing I left unspecified is the kind of modules, and the appropriate smallness condition. Luckily, I can show that $\mathrm{K}(n+1)$-localized K-theory doesn't see the difference between any reasonable choice. Nevertheless, a particularly natural choice would be the atomic $\mathrm{K}(n)$-local modules, one reason being that higher semiadditivity also implies that taking atomics commutes with $(-)^X$.

Finally, it remains to explain the compatibility of the various comparisons. As I have mentioned but glossed over, there is a relationship between $\mathrm{E}_{n+1}^{t\mathbb{Z}/p^r}$ and C_n. In fact, I prove a stronger result: the transchromatic character map is the canonical map to the *proper* Tate construction. I believe this will be key to the comparison (and has other applications in its own right).