

Categorical Ambidexterity [BM26]

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1 Introduction

Limits of categories¹ are generally speaking approachable. E.g., an object in such a limit is a compatible system of objects in category. Moreover, it doesn't matter in which ambient category of categories one computes the limits, i.e. the forgetful functors from Cat_{st} , $\text{Cat}_{\mathcal{K}}$, Pr^{L} , Pr^{R} , $\text{CMon}(\text{Cat})$, etc. to Cat all preserve limits. On the other hand, colimits are generally much more difficult to compute. Let me highlight a special phenomenon.

Theorem 1 (Lurie [Lur17]). *Let $\mathcal{C}_{\bullet}: X \rightarrow \text{Pr}^{\text{L}}$ be a diagram indexed by $X \in \mathcal{S}$, then*

$$\text{colim}_X \mathcal{C}_{\bullet} \simeq \lim_X \mathcal{C}_{\bullet}.$$

Lurie's proof of this result (which in fact holds indexed by a category up to an op) relies on the very special phenomenon that $\text{Pr}^{\text{L}} \simeq (\text{Pr}^{\text{R}})^{\text{op}}$ by passing to right adjoint morphisms, utilizing the fact that both forgetful functors preserve limits.

Example 2. This theorem is non-trivial even in the case of a constant diagram $\mathcal{S}: X \rightarrow \text{Pr}^{\text{L}}$, where it translates to the fact that the presheaf category is the free cocompletion

$$\text{Fun}^{\text{L}}(\mathcal{P}(X), \mathcal{D}) \simeq \text{Fun}^{\text{L}}(\lim_X \mathcal{S}, \mathcal{D}) \simeq \text{Fun}^{\text{L}}(\text{colim}_X \mathcal{S}, \mathcal{D}) \simeq \lim_X \text{Fun}^{\text{L}}(\mathcal{S}, \mathcal{D}) \simeq \lim_X \mathcal{D} \simeq \text{Fun}(X, \mathcal{D}).$$

Yonatan Harpaz proved a strikingly similar theorem. Recall that a space is called π -finite if it is truncated and all homotopy groups are finite. Let $\text{Cat}_{\pi\text{-fin}}$ denote the category of categories admitting π -finite colimits and functors preserving them.

Theorem 3 (Harpaz [Har20]). *Let $\mathcal{C}_{\bullet}: X \rightarrow \text{Cat}_{\pi\text{-fin}}$ be a diagram indexed by $X \in \mathcal{S}_{\pi\text{-fin}}$, then*

$$\text{colim}_X \mathcal{C}_{\bullet} \simeq \lim_X \mathcal{C}_{\bullet}.$$

Yonatan's proof is completely different from the aforementioned proof, and relies on his universal property of $\text{Span}(\mathcal{S}_{\pi\text{-fin}})$.

Remark 4. My personal interest in this theorem is its role in redshift. Indeed, it allows us to prove — or even formulate — the semiadditive redshift theorem [CSY21; BMS24].

¹As usual, in this talk category means $(\infty, 1)$ -category and n -category means (∞, n) -category.

The strikingly similarity between these theorem led me to wonder whether they admit a common generalization and a uniform proof, which I'd like to explain.

Notation 5. We fix a collection indexing spaces $\mathcal{K}_0 \subseteq \mathcal{S}$ which contains pt and is closed under pullbacks, and some indexing categories $\mathcal{K}_0 \subseteq \mathcal{K} \subseteq \text{Cat}$. Let $\text{Cat}_{\mathcal{K}}$ denote the category of categories admitting \mathcal{K} -colimits and functors preserving them.

Theorem 6 (B.-M.). *Let $\mathcal{C}_{\bullet}: X \rightarrow \text{Cat}_{\mathcal{K}}$ be a diagram indexed by $X \in \mathcal{K}_0$, then*

$$\text{colim}_X \mathcal{C}_{\bullet} \simeq \lim_X \mathcal{C}_{\bullet}.$$

Example 7. Taking $\mathcal{K}_0 = \mathcal{K} = \mathcal{S}_{\pi\text{-fin}}$ implies theorem 3. Taking $\mathcal{K}_0 = \mathcal{S}$ and $\mathcal{K} = \text{Cat}$ implies theorem 1, and in fact shows that it already holds in cocomplete categories (or even those admitting \mathcal{S} -colimits) where we don't have access to the right adjoints.

As above, even the case of a constant diagram is non-trivial, giving $\mathcal{C}[X] \simeq \mathcal{C}^X$. Furthermore, observe that $\mathcal{C}[-]$ is covariantly functorial in $f: X \rightarrow Y$, while $\mathcal{C}^{(-)}$ is contravariantly functorial for pullbacks f^* . Note though that since \mathcal{C} has \mathcal{K} -colimits, the pullback has a left adjoint given by left Kan extension, and I prove the following.

Theorem 8 (B.-M.). *For $\mathcal{C} \in \text{Cat}_{\mathcal{K}}$ there is an isomorphism*

$$\mathcal{C}[X] \simeq \mathcal{C}^X$$

natural in $X \in \mathcal{K}_0$ for the left Kan extension functoriality on the right-hand side.

The plan of the proof is somewhat surprising, following these steps:

- (1) theorem 8 for $\mathcal{C} = \mathcal{S}_{\mathcal{K}}$,
- (2) theorem 8 for general \mathcal{C} ,
- (3) a stronger theorem that implies both – surprisingly this uses 3-category theory,
- (4) deduce theorem 6.

2 Proof of constant case

Recall that the forgetful admits a left adjoint $\mathcal{P}^{\mathcal{K}}: \text{Cat}_{\mathcal{K}} \rightarrow \text{Cat}$, freely adjoining \mathcal{K} -colimits.

Proposition 9. *There is a natural isomorphism $\mathcal{S}_{\mathcal{K}}[X] \simeq \mathcal{P}^{\mathcal{K}}(X) \simeq \mathcal{S}_{\mathcal{K}}^X$.*

Proof sketch. The first isomorphism holds since both functor commute with colimits and agree on $X = \text{pt}$, and this is clearly natural in X .

For the second isomorphism, recall that from the case of Pr^{L} we have a natural² isomorphism $\mathcal{P}(X) \simeq \mathcal{S}^X$. It remains to show that for $X \in \mathcal{K}_0$, this restricts to an equivalence of the

²In fact the naturality of this is subtle, and is equivalent to the naturality of the Yoneda embedding, first proven by Haugseng–Hebestreit–Linskens–Nuiten [Hau+23].

full subcategories $\mathcal{P}^{\mathcal{K}}(X) \simeq \mathcal{S}_{\mathcal{K}}^X$. Recall that $\mathcal{P}^{\mathcal{K}}(X)$ is the subcategory of those presheaves generated under \mathcal{K} -colimits by the representables $\{\mathcal{Y}(x)\}$, and we need to show these are precisely the presheaves taking values in $\mathcal{S}_{\mathcal{K}}$. To see that they take value in $\mathcal{S}_{\mathcal{K}}$, observe that $\mathcal{Y}(x)(y)$ is the path space $P_{xy}X$ which by closure under pullbacks is in \mathcal{K}_0 , and $\mathcal{S}_{\mathcal{K}}$ is closed under \mathcal{K} -colimits. To show that we get all such presheaves, use the Yoneda density theorem $F \simeq \text{colim}^F \mathcal{Y}$. \square

The following key observation allows us to prove the general constant case.

Observation 10. $\mathcal{S}_{\mathcal{K}}[X]$ is self-dual in $\text{Cat}_{\mathcal{K}}$ (since $\mathcal{S}_{\mathcal{K}}[X] \simeq \mathcal{S}_{\mathcal{K}}^X$ mates to a perfect pairing).

Proposition 11. *There is a natural isomorphism $\mathcal{C}[X] \simeq \mathcal{C}^X$.*

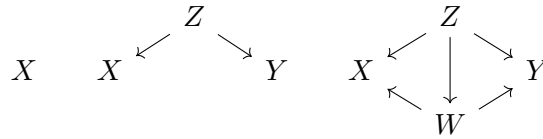
Proof idea. The naturality is in fact quite subtle, and I will not comment on it. Pointwise this is easy given observation 10:

$$\mathcal{C}[X] \simeq \mathcal{S}_{\mathcal{K}}[X] \otimes \mathcal{C} \simeq \mathcal{S}_{\mathcal{K}}[X]^{\vee} \otimes \mathcal{C} \simeq \text{Fun}_{\mathcal{K}}(\mathcal{S}_{\mathcal{K}}[X], \mathcal{C}) \simeq \mathcal{C}^X.$$

\square

3 Spans

Let \mathcal{X} be any category closed under pullbacks, which we will later take to be \mathcal{K}_0 . Consider the 2-category $\mathbf{Span}_{1\frac{1}{2}}(\mathcal{X})$ with objects, 1-morphisms and 2-morphisms depicted below.



It is a well-known observation that the 2-morphisms encode adjunction data, making the 1-morphism $X = X \rightarrow Y$ left adjoint to $Y \leftarrow X = X$.³ In fact, this characterizes the 2-category.

Theorem 12 ([GR19; Mac22; Ste20]). $\mathcal{X}^{\text{op}} \rightarrow \mathbf{Span}_{1\frac{1}{2}}(\mathcal{X})$ is the universal 2-category under \mathcal{X}^{op} where every morphism is sent to a right adjoint morphism, and satisfying the Beck–Chevalley condition.

Example 13. Let $\mathcal{X} \subseteq \mathcal{S}$, let \mathcal{D} be a category with \mathcal{X} -colimits, and consider $\mathcal{D}^{(-)} : \mathcal{X}^{\text{op}} \rightarrow \mathbf{Cat}$. The colimits assumption says that $f^* : \mathcal{D}^Y \rightarrow \mathcal{D}^X$ has a left adjoint $f_!$, given by left Kan extension.⁴ We conclude that there is a unique lift

$$\mathcal{D}^{(-)} : \mathbf{Span}_{1\frac{1}{2}}(\mathcal{X}) \longrightarrow \mathbf{Cat}.$$

Dually, if \mathcal{D} has \mathcal{X} -limits, then we have $f^* \dashv f_*$, and we get

$$\mathcal{D}^{(-)} : \mathbf{Span}_{1\frac{1}{2}}(\mathcal{X})^{2\text{-op}} \longrightarrow \mathbf{Cat}.$$

³Hint: form the two compositions by forming pullbacks, and construct the unit and counit using $X \rightarrow Y$ and the relative diagonal $X \rightarrow X \times_Y X$.

⁴The Beck–Chevalley condition translates to the formula for left Kan extensions.

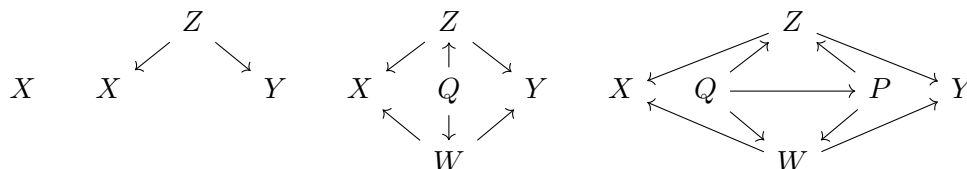
Subexample 14. Let $\mathcal{C} \in \mathbf{Cat}_{\mathcal{K}}$, then $\mathcal{C}^{(-)}: \mathcal{S}_{\mathcal{K}}^{\text{op}} \rightarrow \mathbf{Cat}$ lifts to $\mathbf{Span}_{1\frac{1}{2}}(\mathcal{K}_0)$.

Subexample 15. All categories of categories we imagine, such as $\mathcal{D} = \mathbf{Cat}_{\mathcal{K}}$, are complete and cocomplete, so they are examples for $\mathcal{X} = \mathcal{S}$, giving us

$$\mathbf{Cat}_{\mathcal{K}}^{(-)}: \mathbf{Span}_{1\frac{1}{2}}(\mathcal{S}) \longrightarrow \mathbf{Cat}, \quad \mathbf{Cat}_{\mathcal{K}}^{(-)}: \mathbf{Span}_{1\frac{1}{2}}(\mathcal{S})^{2\text{-op}} \longrightarrow \mathbf{Cat}.$$

Observe that theorem 6 (which we haven't proven yet) asserts that for $f: X \rightarrow \text{pt}$ with $X \in \mathcal{K}_0$, the two adjoints of f^* canonically agree $f_! \simeq f_*$. Similarly theorem 8 (whose proof I sketched above) says that they agree on constant diagrams for any $f: X \rightarrow Y$ in \mathcal{K}_0 . We see that upon restricting from \mathcal{S} to \mathcal{K}_0 , the lifts to $\mathbf{Span}_{1\frac{1}{2}}(\mathcal{S})$ and $\mathbf{Span}_{1\frac{1}{2}}(\mathcal{S})^{2\text{-op}}$ should somehow be compatible.

We are thus naturally led to the idea of finding some categorical gadget which contains both $\mathbf{Span}_{1\frac{1}{2}}(\mathcal{X})$ and $\mathbf{Span}_{1\frac{1}{2}}(\mathcal{X})^{2\text{-op}}$, plus isomorphisms $f_! \simeq f_*$. The isomorphisms look like extra data, but if done correctly they aren't. Consider the 3-category $\mathbf{Span}_{2\frac{1}{2}}(\mathcal{X})$ with objects, 1-morphisms, 2-morphisms and 3-morphisms depicted below.



Note that by picking the 2-morphisms that go downwards or upwards, both $\mathbf{Span}_{1\frac{1}{2}}(\mathcal{X})$ and $\mathbf{Span}_{1\frac{1}{2}}(\mathcal{X})^{2\text{-op}}$ include into $\mathbf{Span}_{2\frac{1}{2}}(\mathcal{X})$. Consequently, by the universal property above, $\mathbf{Span}_{2\frac{1}{2}}(\mathcal{X})$ contains adjunction data for the 1-morphisms $X = X \rightarrow Y$ and $Y \leftarrow X = X$ from *both sides*, as we'd like. This of course does not automatically mean that mapping out of $\mathbf{Span}_{2\frac{1}{2}}(\mathcal{X})$ has a universal property, since this may a priori be extra structure.⁵ Nevertheless, this is the case. In $\mathbf{Span}_{1\frac{1}{2}}(\mathcal{X})$, the 2-morphisms encoded that the two legs are adjoint. Similarly, the 3-morphisms in $\mathbf{Span}_{2\frac{1}{2}}(\mathcal{X})$ encode a compatibility between the unit and counit of the two sided adjunction, which makes their identification into a condition rather than structure, as I will explain below. Stefanich proves that this completely characterizes it.

Theorem 16 (Stefanich [Ste20]). $\mathcal{X}^{\text{op}} \rightarrow \mathbf{Span}_{2\frac{1}{2}}(\mathcal{X})$ is the universal 3-category under \mathcal{X}^{op} where every morphism is sent to a morphism that is both a right and a left adjoint, and satisfying the 2-fold Beck–Chevalley condition.

I did not specify the the 2-fold Beck–Chevalley condition. Let me exemplify a core part, which is both (hopefully) illuminating and (unfortunately) confusing. Let $\mathcal{D}(-): \mathcal{X}^{\text{op}} \rightarrow \mathbf{A}$ be a functor into some 3-category and let $f: X \rightarrow \text{pt}$. The adjunction

$$f_!: \mathcal{D}(X) \rightleftarrows \mathcal{D}(\text{pt}): f^*$$

⁵Indeed, we can for example consider the maximal 2-category $\mathbf{Span}_2(\mathcal{X})$ which already contains these adjunction data, but does not in general have a mapping out universal property. Surprisingly it does in the truncated case, as I conjectured and was later proven by Cnossen–Lenz–Linskens.

has a counit $c_f: f_! f^* \rightarrow \text{Id}_{\mathcal{D}(\text{pt})}$, living in the 2-category $\text{hom}(\mathcal{D}(\text{pt}), \mathcal{D}(\text{pt}))$. Part of the 2-fold Beck–Chevalley condition is that c_f is itself a left adjoint. The right adjoint $\text{Id}_{\mathcal{D}(\text{pt})} \rightarrow f_! f^*$ is going to be the unit of the adjunction exhibiting $f_!$ as the *right* adjoint of f^* . This in particular explains the mechanism by which the adjunction from both sides is a condition rather than extra information, and also why a 3-category is needed.

4 Strong version

Theorem 17 (B.-M.). *The functor $\mathbf{Cat}_{\mathcal{K}}^{(-)}: \mathcal{K}_0^{\text{op}} \rightarrow \mathbf{Cat}_2$ lifts to 3-functor*

$$\mathbf{Cat}_{\mathcal{K}}^{(-)}: \mathbf{Span}_{2\frac{1}{2}}(\mathcal{K}_0) \longrightarrow \mathbf{Cat}_2.$$

Proof idea. We need to verify the 2-fold Beck–Chevalley condition for $\mathcal{D}(-) = \mathbf{Cat}_{\mathcal{K}}^{(-)}$. Let’s address (part of) the special case I mentioned above. Note that $c_f: f_! f^* \rightarrow \text{Id}_{\mathbf{Cat}_{\mathcal{K}}}$, evaluated at some \mathcal{C} , is simply the fold map $\mathcal{C}[X] \rightarrow \mathcal{C}$. The constant case (theorem 8) identifies $\mathcal{C}[X] \simeq \mathcal{C}^X$. Moreover, it is natural in X for the left Kan extension functoriality, so that the fold map is identified with $\text{colim}_X: \mathcal{C}^X \rightarrow \mathcal{C}$. This is indeed a left adjoint. \square

5 Implication of original case

Finally, this also implies the original case.

Proposition 18. *There is an equivalence $\text{colim}_X \simeq \text{lim}_X$.*

Proof. Consider the span $\text{pt} \leftarrow X = X$. The 3-functor $\mathbf{Cat}_{\mathcal{K}}^{(-)}: \mathbf{Span}_{2\frac{1}{2}}(\mathcal{K}_0) \rightarrow \mathbf{Cat}_2$ sends it to the constant diagram functor $\mathbf{Cat}_{\mathcal{K}} \rightarrow \mathbf{Cat}_{\mathcal{K}}^X$. Recall that in $\mathbf{Span}_{2\frac{1}{2}}(\mathcal{K}_0)$, this span is adjoint to $X = X \rightarrow \text{pt}$ from both sides. Since the 3-functor preserves adjunction data, we conclude that two adjoints colim_X are lim_X are identified. \square

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