Variations on rotation invariance in K-theory

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Plan

- Commutativity and strict commutativity
- Strict triviality in algebraic K-theory
- Rotation invariance and the proof

- Joint with Qingyuan Bai

Commutativity and strict commutativity

Commutativity in ordinary algebra

- For elements of a commutative monoid $x, y \in A$

$$x \cdot y = y \cdot x$$

- In particular, for $x = y$

$$x \cdot x = x \cdot x$$

nothing much to say

Commutativity in higher algebra

- For objects in a category $X, Y \in \mathcal{C}$

$$B_{X,Y} \colon X \times Y \xrightarrow{\simeq} Y \times X, \qquad B_{X,Y}(x, y) = (y, x)$$

- In particular, for $X = Y$

$$B_{X,X} \colon X \times X \xrightarrow{\simeq} X \times X, \qquad B_{X,X}(x_1, x_2) = (x_2, x_1)$$

- More generally, Σ_n acts on X^n

Commutativity in higher algebra (cont.)

- Given $(A, \otimes) \in \mathrm{CMon}(\mathcal{S})$ (commutative monoid structure on a space)

  for $X \in A$ and $n \in \mathbb{N}$ have

$$\Sigma_n \curvearrowright X^{\otimes n} \qquad \in \qquad A$$

- Saying $X \in A$ “commutes with itself” is extra structure: trivialization of $\Sigma_n \curvearrowright X^{\otimes n}$

Strict commutativity

- Element $X \in A \quad \iff \quad$ map $\mathrm{Fin}^{\simeq} \to A \ \in \ \mathrm{CMon}(\mathcal{S})$

  encoding all Σ_n-action via

$$\mathrm{Fin}^{\simeq} \simeq \bigsqcup \mathrm{B}\Sigma_n$$

Definition

The structure of a *strict element* on $X \in A$ is a factorization

$$\begin{tikzcd} \mathrm{Fin}^{\simeq} \arrow[r, "X"] \arrow[d] & A \\ \mathbb{N} \arrow[ur, dashed, bend right] & \end{tikzcd} \quad \in \quad \mathrm{CMon}(\mathcal{S})$$

Strict commutativity (cont.)

- Distinction does not exist in ordinary algebra
  $\mathbb{N}$ is both the free monoid and free commutative monoid
  (note false for free on 2 generators)
- $\mathbb{N}$ is the free monoid in spaces, $\mathrm{Fin}^{\simeq}$ is the free commutative monoid
- Alternative definition:
  Element $X \in A \quad \iff \quad$ *monoid* map $\mathbb{N} \to A$,
  strict structure is a *commutative* monoid lift

Examples

Example

The neutral element $\mathbf{1} \in A$ canonically lifts to a strict element

Example

The only strict sets are $\emptyset$ and $*$
(consider the swap $B_{X,X} \colon X \times X \xrightarrow{\sim} X \times X$)

Example

Any line bundle over a commutative ring R (module M such that $M_{\mathfrak{p}} \simeq R_{\mathfrak{p}}$ for all $\mathfrak{p} \lhd R$) is canonically strict
(since locally on $\operatorname{Spec}(R)$ it is the unit)

Derived Modules

- For a R commutative ring, consider

$$\mathcal{D}(R) = \mathrm{Ch}(\mathrm{Mod}_R)[\mathrm{q.iso}^{-1}] = \mathrm{Mod}_R(\mathrm{Sp})$$

  with the (derived) tensor product $\otimes_R$
- By previous example, line bundles over R are strict

Question

What about shifts $R[n]$?

Proposition

$R[n] \in \mathcal{D}(R)$ *is strict if* n *is even, and isn't (in general) if* n *is odd*

Strict derived modules

Proposition

$R[n] \in \mathcal{D}(R)$ is strict if n is even, and isn't (in general) if n is odd

Proof.

Suffices for $R = \mathbb{Z}$.
Consider the groupoid $\mathrm{pic}(\mathbb{Z})$ on $\{\mathbb{Z}[i]\}_i$,
has $\pi_0 = \mathbb{Z}$ and $\pi_1 = \mathbb{Z}^\times = \mathbb{Z}/2$, and k-invariant by Sq^2.

$$\begin{tikzcd}
\mathbb{N} \arrow[dr, dashed] \arrow[drrr, bend left] \arrow[ddr, bend right, "n"'] & & & \\
& \mathrm{pic}(\mathbb{Z}) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & 0 \arrow[d] \\
& \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r, "\mathrm{Sq}^2"] & \mathbb{Z}/2[2]
\end{tikzcd}$$

□

Summary on commutativity

- Elements have non-trivial actions

$$\Sigma_n \curvearrowright X^{\otimes n} \qquad \in \qquad A$$

- Trivialization is extra data

$$\mathbb{N} \longrightarrow A \qquad \in \qquad \mathrm{CMon}(\mathcal{S})$$

- $R[2] \in \mathcal{D}(R)$ is strict, but $R[1]$ isn't

Strict triviality in algebraic K-theory

Algebraic K-theory

- Consider perfect modules: bounded complexes of finitely generated projective (vector bundles)

$$\mathrm{Perf}(R) := \mathrm{Ch}^b(\mathrm{Proj}_R)[\mathrm{q.iso}^{-1}] = \mathcal{D}(R)^{\mathrm{dbl}}$$

- $\mathrm{K}(R)$ obtained from $\mathrm{Perf}(R)^{\simeq}$ by splitting exact sequences

$$X \longrightarrow Y \longrightarrow Z \qquad \rightsquigarrow \qquad [X] + [Z] \simeq [Y]$$

- Multiplicative structure

$$[X] \cdot [Y] = [X \otimes_R Y]$$

Triviality of $[R[2]]$

- $X[1]$ participates in

$$X \longrightarrow 0 \longrightarrow X[1] \qquad \rightsquigarrow \qquad [X[1]] \simeq -[X]$$

- Applied twice to $X = R$

$$[R[2]] \simeq [R]$$

i.e. $[R[2]]$ is the unit element in $\mathrm{K}(R)$

Question

Is $[R[2]] \simeq [R]$ a "coincidence", or does it hold more broadly?

Strict triviality of $[R[2]]$

- Recall the unit, $[R] \in \mathrm{K}(R)$, is always trivially strict
- Produced strict structure on $R[2] \in \mathcal{D}(R)$
  $\rightsquigarrow$ some strict structure on $[R[2]] \in \mathrm{K}(R)$

Theorem (Bai–B.-M., in progress)

The strict structure on $[R[2]]$ is trivial, i.e. the composition

$$\mathbb{N} \longrightarrow \mathrm{Perf}(R)^{\simeq} \longrightarrow \mathrm{K}(R)$$

is null-homotopic

Application

Theorem (Carmeli–Luecke (2024))

The second k-invariant of $\mathrm{K}(\mathbb{Z})^\times$ vanishes, i.e.

$$\mathrm{K}(\mathbb{Z})^\times \simeq \tau_{\leq 1}\,\mathrm{K}(\mathbb{Z})^\times \oplus \tau_{\geq 2}\,\mathrm{K}(\mathbb{Z})^\times$$

($\rightsquigarrow$ same for $\mathrm{K}(-)^\times$, after Zariski sheafification)

- By sophisticated analysis and chromatic methods

Alternative proof.

$\mathrm{pic}(\mathbb{Z}) \to \mathrm{K}(\mathbb{Z})^\times$ factors through the quotient by $\mathbb{Z}[2]$, yielding an isomorphism onto $\tau_{\leq 1}\,\mathrm{K}(\mathbb{Z})^\times$, hence a section ☐

Rotation invariance and the proof

Spectra

- So far considered $\mathcal{D}(R)$ for ordinary R
  - Initial commutative semiring: $(\mathbb{N}, +, \cdot)$
  - Initial commutative ring: $\mathbb{Z} = \mathbb{N}^{\mathrm{gpc}}$
- Consider commutative rings in spaces
  - Initial commutative semiring in spaces: $(\mathrm{Fin}^{\simeq}, \sqcup, \times)$
  - Initial commutative ring in spaces: $\mathbb{S} = (\mathrm{Fin}^{\simeq})^{\mathrm{gpc}}$
- Universal example $\mathrm{Sp} = \mathcal{D}(\mathbb{S})$

Strictness in spectra

- Refinement: have $\mathbb{S} \to \mathbb{Z}$ inducing $\mathrm{Sp} \to \mathcal{D}(\mathbb{Z})$
- $\mathbb{Z}[1]$ isn't strict $\quad\rightsquigarrow\quad$ $\mathbb{S}[1]$ isn't strict

Question

Does the strict structure on $\mathbb{Z}[2]$ lift to $\mathbb{S}[2]$?

Theorem (Lurie (2015))

There is an $\mathbb{E}_2$-lift of $\mathbb{N} \to \mathrm{Sp}$ choosing $\mathbb{S}[2]$
(No $\mathbb{E}_3$-lift, in particular, no strict lift)

- Two ways: combinatorially, and via Bott periodicity

Rotation

- Shift by 2 on any stable ∞-category

  $$(-)[2] \colon \mathcal{C} \longrightarrow \mathcal{C}$$

  is invertible, and natural in $\mathcal{C}$
- $\mathbb{Z} \to \mathrm{Aut}(\mathrm{Id}_{\mathrm{Cat}_{\mathrm{st}}})$ monoid map
- Lurie's theorem makes it $\mathbb{E}_2$
- "Rotation action", deloop $S^1 = \mathrm{B}\mathbb{Z} \to \mathrm{Aut}(\mathrm{Cat}_{\mathrm{st}})$

Rotation invariance

Theorem (Lurie (2015))

The functor $\mathrm{K}\colon \mathrm{Cat}_{\mathrm{st}} \to \mathrm{Sp}$ *is invariant under the* S^1*-action*

- “Self-commuting” upgrade of the fact $[X[2]] \simeq [X]$
- Using combinatorics of paracyclic $S_\bullet$-construction

- Subtle point: rotation invariance a priori *weaker* than $\mathbb{E}_2$-lift of $\mathbb{S}[2]$ (doesn’t use multiplicativity of K)

Theorem (Bai–B.-M., in progress)

The $\mathbb{E}_2$*-structure on* $[\mathbb{S}[2]]$ *is trivial, i.e. the* $\mathbb{E}_2$*-map*

$$\mathbb{N} \longrightarrow \mathrm{Perf}(\mathbb{S})^{\simeq} \longrightarrow \mathrm{K}(\mathbb{S})$$

is null-homotopic

Strict rotation invariance

- Recall $\mathbb{Z}[2]$ is strict ($\mathbb{E}_\infty$-lift, not just $\mathbb{E}_2$), want

Theorem (Bai–B.-M., in progress)

The strict structure on $[R[2]]$ *is trivial, i.e. the composition*

$$\mathbb{N} \longrightarrow \mathrm{Perf}(R)^{\simeq} \longrightarrow \mathrm{K}(R)$$

is null-homotopic

- Use combinatorial description of multiplicative structure
- Prove holds over MU, using Bott periodicity description
- Prove holds for any localizing invariant $(\mathrm{THH}, \mathrm{TC}, \ldots)$

Thank You!