

Algebraic K-theory and higher semiadditivity

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I'd like to present some ideas about redshift from a higher semiadditive perspective. In the first part of the talk, I want to present an application of the redshift philosophy, to give a proof of higher semiadditivity. In the second part, I hope to clarify in a way why this works, through the semiadditive redshift theorem, and then also present some related work on higher semiadditive K-theory.

1 Higher semiadditivity by redshift

Higher semiadditivity was originally proved for $\mathrm{Sp}_{K(n)}$ by Hopkins–Lurie, and later proved for all of $\mathrm{Sp}_{T(n)}$ by Carmeli–Schlank–Yanovski. In this part I want to explain a proof of higher semiadditivity by induction on the height, using algebraic K-theory, proved simultaneously with $T(n+1)$ -localized K-theory being higher semiadditive. Let me point out that the latter is only p -typically ∞ -semiadditive (indeed, it doesn't commute with general finite group actions) so throughout I use the term ∞ -semiadditive for p -typical ∞ -semiadditive. We shall carry out the inductive step going from n to $n+1$ (this argument only works for $n \geq 1$, so heights 0 and 1 need to be dealt with separately).

Assumption 1. $\mathrm{Sp}_{K(n)}$ is ∞ -semiadditive.

To start, let me recall that we say that $\mathcal{C}$ is ∞ -semiadditive if for every diagram $X_\bullet: A \rightarrow \mathcal{C}$ indexed by a π -finite p -space, the (inductively defined) norm map is an isomorphism

$$\mathrm{Nm}: \operatorname{colim}_A X_\bullet \xrightarrow{\sim} \lim_A X_\bullet.$$

We can reduce to the case of constant diagrams, that is $X[A] \xrightarrow{\sim} X^A$, by a retract argument. In the monoidal case, we can further reduce to testing this on the unit $1_{\mathcal{C}}[A] \xrightarrow{\sim} 1_{\mathcal{C}}^A$, or equivalently, that

$$1_{\mathcal{C}}[A] \otimes 1_{\mathcal{C}}[A] \longrightarrow 1_{\mathcal{C}}$$

is a perfect pairing. This self-duality is preserved under quite minimal assumptions.

Lemma 2. *Let $\mathcal{C}, \mathcal{D}$ be monoidal, and assume that the tensor product commutes with π -finite p -space colimits. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a lax monoidal functor, and assume that it is unital and for every π -finite p -space the assembly map*

$$F(1_{\mathcal{C}}[A]) \longrightarrow F(1_{\mathcal{C}}^A)$$

is an isomorphism. If $\mathcal{C}$ is ∞ -semiadditive then so are $\mathcal{D}$ and F .

We would like to apply this to $F = K_{T(n+1)}$. The first question that arises is what the source should be. Notice that while $\mathrm{Sp}_{K(n)}$ is ∞ -semiadditive, there's no chance for $\mathrm{Alg}(\mathrm{Sp}_{K(n)})$ to be ∞ -semiadditive, so we should take K-theory from some category of categories.

Theorem 3 (Harpaz, B.M.). *The category $\mathrm{Cat}_{p\text{-fin}}$ is ∞ -semiadditive.*

Proof idea. This is, in fact, a very general statement about $\mathcal{K}$ -limits and colimits in $\mathrm{Cat}_{\mathcal{K}}$ for any collection of shapes $\mathcal{K} \subset \mathcal{S}$ closed under finite limits. As above, we can check on the unit $\mathcal{S}_{\mathcal{K}}$. One first observes that presheaf category is both described as functors out of A , and as the free cocompletion. Then, one checks that these restrict accordingly to the subcategories.

$$\begin{array}{ccccc} \mathcal{S}[A] & \xleftarrow{\sim} & \mathcal{P}(A) & \xrightarrow{\sim} & \mathcal{S}^A \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{S}_{\mathcal{K}}[A] & \xleftarrow{\sim} & \mathcal{P}^{\mathcal{K}}(A) & \xrightarrow{\sim} & \mathcal{S}_{\mathcal{K}}^A \end{array}$$

One also needs to check that the resulting map is the norm map, which must be true by the unique map principle (though in practice my proof uses $(\infty, 3)$ -categories). $\square$

Using the fact that π -finite p -space colimits are self-dual in an ∞ -semiadditive category, we thus have the following example (under [Assumption 1](#)).

Proposition 4 (B.M.–Schlank). *$\mathrm{Dbl}(R) := \mathrm{Mod}_R(\mathrm{Sp}_{K(n)})^{\mathrm{dbl}}$ is in $\mathrm{Cat}_{p\text{-fin}}^{\mathrm{st}}$ for $R \in \mathrm{Alg}(\mathrm{Sp}_{K(n)})$.*

I would like to point out that for the purposes of chromatically localized K-theory it doesn't matter whether one works with $\mathrm{Dbl}(R)$ or $\mathrm{Perf}(R)$.

Proposition 5 (Clausen–Mathew–Naumann–Noel). *The inclusion $\mathrm{Perf}(R) \hookrightarrow \mathrm{Dbl}(R)$ induces an isomorphism*

$$K_{T(n+1)}(R) = K_{T(n+1)}(\mathrm{Perf}(R)) \xrightarrow{\sim} K_{T(n+1)}(\mathrm{Dbl}(R)).$$

Proof. The Verdier quotient $\mathrm{Dbl}(R)/\mathrm{Perf}(R)$ is L_{n-1}^f -linear, so has vanishing $K_{T(n+1)}$ by the red-shift upper bound due to Clausen–Mathew–Naumann–Noel (to which I recently gave another proof). The result thus follows from the fact that K-theory is localizing. $\square$

Thus, we can restrict to modules over $\mathrm{Dbl}(R)$, e.g. $\mathrm{Dbl}(\mathbb{S}_{K(n)}) = \mathrm{Sp}_{K(n)}^{\mathrm{dbl}}$, that is consider

$$K_{T(n+1)} : \mathrm{Mod}_{\mathrm{Dbl}(R)}(\mathrm{Cat}_{p\text{-fin}}^{\mathrm{st}}) \longrightarrow \mathrm{Mod}_{K_{T(n+1)}(R)}(\mathrm{Sp}_{T(n+1)}),$$

and we have the following.

Proposition 6 (B.M.–Carmeli–Schlank–Yanovski). *The functor above preserves constant π -finite p -space colimits on the unit*

$$K_{T(n+1)}(\mathrm{Dbl}(R))[A] \xrightarrow{\sim} K_{T(n+1)}(\mathrm{Dbl}(R)[A]).$$

Proof. We argue by induction on the truncatedness level of A . If A is a finite set, this is just the commutativity of K-theory with products. We may henceforth assume A is connected. If $A = BG$ is the classifying space of a finite p -group, this is an instance of the descent theorem of Clausen–Mathew–Naumann–Noel. Thus we may also assume from now on A is simply connected.

Assume A is m -truncated. Recall that we assume that A is connected, and using ∞ -semiadditivity we can show

$$\mathrm{Dbl}(R)[A] \simeq \mathrm{Dbl}(R)^A \simeq \mathrm{Dbl}(R[\Omega A]).$$

Thus we can write the map in question as the top map below, which is also equivalent to the bottom map by [Proposition 5](#) (together with the purity theorem to see that it doesn't matter whether the group algebra is $K(n)$ -localized or not)

$$\begin{array}{ccc} K_{T(n+1)}(\mathrm{Dbl}(R))[A] & \longrightarrow & K_{T(n+1)}(\mathrm{Dbl}(R[\Omega A])) \\ \uparrow \wr & & \uparrow \wr \\ K_{T(n+1)}(R)[A] & \longrightarrow & K_{T(n+1)}(R[\Omega A]) \end{array}$$

Since A is connected, we can express it via the bar construction $A \simeq \mathrm{colim}_{\Delta^{\mathrm{op}}}(\Omega A)^{\bullet}$. Since A is simply connected, ΩA is connected and so it is the loop of the previous bar construction. The purity theorem of Land–Mathew–Meier–Tamme implies that $K_{T(n+1)}$ commutes with sifted colimits of ring spectra, so everything in sight preserves sifted colimits, so we are reduced to the case of $(\Omega A)^k$, which is $(m-1)$ -truncated, hence holds by inductive hypothesis. $\square$

Theorem 7. *The codomain and the functor below are ∞ -semiadditive*

$$K_{T(n+1)} : \mathrm{Mod}_{\mathrm{Dbl}(R)}(\mathrm{Cat}_{p\text{-fin}}^{\mathrm{st}}) \longrightarrow \mathrm{Mod}_{K_{T(n+1)}(R)}(\mathrm{Sp}_{T(n+1)}).$$

Proof. The functor is unital lax symmetric monoidal, and by [Proposition 6](#) it preserves constant colimits, so we conclude by [Lemma 2](#). $\square$

Theorem 8. *$\mathrm{Sp}_{K(n+1)}$ is ∞ -semiadditive.*

Proof. This is a descendability argument. Notice that the collection Y of spectra such that for every $X_{\bullet} : A \rightarrow \mathrm{Sp}_{K(n+1)}$

$$\mathrm{colim}_A(X_{\bullet} \otimes Y) \xrightarrow{\sim} \lim_A(X_{\bullet} \otimes Y)$$

is a thick $\otimes$ -ideal. By [Theorem 7](#), we know $K_{T(n+1)}(R)$ is inside. We show E_{n+1} is inside. This can be done with nullstellensatz, or more directly, in Yuan's proof of redshift he constructs a ring map

$$K_{T(n+1)}(E_{n+1}^{tC_p}) \longrightarrow L_{T(n+1)}(E_{n+1}^{tS^1}),$$

so the latter is in the thick $\otimes$ -ideal, and since E_{n+1} is complex oriented, it is a retract of $E_{n+1}^{tS^1}$. Finally, E_{n+1} is descendable over $\mathbb{S}_{K(n+1)}$ by the proof of the smash product theorem. $\square$

Of course, it is natural to ask whether one can work telescopically. This would follow from a positive answer to the following, which I believe others have thought a lot about.

Question 9. *Is $K_{T(n+1)}(\mathbb{S}_{T(n)})$ descendable over $\mathbb{S}_{T(n+1)}$?*

2 Semiadditive redshift

One may wonder how did we manage to increase in height, which can (partly) be explained by semiadditive redshift. To put it very briefly, recall that chromatic height is

$$\begin{aligned} \mathrm{ht}^{\mathrm{st}}(X) &\leq n && \text{if } X \text{ is } v_n\text{-invertible} \\ \mathrm{ht}^{\mathrm{st}}(X) &> n && \text{if } X \text{ is } v_n\text{-complete.} \end{aligned}$$

Higher semiadditivity gives rise to the self maps

$$|B^n C_p|: X \longrightarrow X^{B^n C_p} \simeq X[B^n C_p] \longrightarrow X.$$

Carmeli–Schlank–Yanovski developed a theory of semiadditive height, defined by

$$\begin{aligned} \mathrm{ht}^{\oplus}(X) &\leq n && \text{if } X \text{ is } |B^n C_p|\text{-invertible} \\ \mathrm{ht}^{\oplus}(X) &> n && \text{if } X \text{ is } |B^n C_p|\text{-complete.} \end{aligned}$$

Example 10. For $X \in \mathrm{Sp}_{T(n)}$ we have $\mathrm{ht}^{\oplus}(X) = n$.

An important feature of this definition is that it extends beyond the stable setting, to any ∞ -semiadditive category. In particular, we can apply it in $\mathrm{Cat}_{p\text{-fin}}$. Thus, for any ∞ -semiadditive category $\mathcal{C}$, we can both measure the height *in* $\mathcal{C}$ (as above), and *of* $\mathcal{C}$ (in $\mathrm{Cat}_{p\text{-fin}}$), and we have the following manifestation of the redshift philosophy in the categorical setting, prior to applying K-theory.

Theorem 11. *Let $\mathcal{C}$ be an ∞ -semiadditive category, and $n \in \mathbb{N}$, then the following are equivalent*

- (1) $\mathrm{ht}^{\oplus}(X) \leq n$ for all $X \in \mathcal{C}$,
- (2) $\mathrm{ht}^{\oplus}(\mathcal{C}) \leq n+1$,

and the same holds for $>$.

Example 12. Let $R \in \mathrm{Alg}(\mathrm{Sp}_{T(n)})$, then $\mathrm{ht}^{\oplus}(\mathrm{Dbl}(R)) = n+1$.

Therefore, we see explicitly that the increase in height in the previous section is produced by categorification, where we moved from $\mathrm{Sp}_{K(n)}$, through $\mathrm{Mod}_{\mathrm{Dbl}(\mathbb{S}_{K(n)})}(\mathrm{Cat}_{p\text{-fin}}^{\mathrm{st}})$ whose objects are of height $n+1$, and then to $\mathrm{Sp}_{K(n+1)}$ using the ∞ -semiadditivity of $K_{T(n+1)}$.

3 Semiadditive K-theory

We saw that $\mathrm{Cat}_{p\text{-fin}}^{\mathrm{st}}$ is ∞ -semiadditive. Note that K-theory is defined on all such categories, but we, somewhat artificially, restricted to modules over $\mathrm{Dbl}(\mathbb{S}_{K(n)})$, and $T(n+1)$ -localized the

target, getting ∞ -semiadditivity. It is natural to try to abstract these, and consider the universal ∞ -semiadditive variant of K-theory defined on $\text{Cat}_{p\text{-fin}}^{\text{st}}$. To make sense of this, the target must be ∞ -semiadditive, so we need to universally make spectra ∞ -semiadditive.

Note that if $X \in \mathcal{C}$ is an object in an ∞ -semiadditive category, then (co)homology is both covariantly and contravariantly functorial for maps of π -finite spaces

$$A \leftarrow B \rightarrow C \quad \rightsquigarrow \quad X^A \rightarrow X^B \simeq X[B] \rightarrow X[C].$$

It turns that this is exactly characterizes ∞ -semiadditivity.

Theorem 13 (Harpaz). *The universal ∞ -semiadditive category under $\mathcal{C} \in \text{Pr}^{\text{L}}$ is*

$$\mathcal{C} \longrightarrow \text{CMon}_{\infty}(\mathcal{C}) := \text{Fun}^{\text{seg}}(\text{Span}(\mathcal{S}_{p\text{-fin}}), \mathcal{C}).$$

Remark 14. In particular, $\mathcal{C}$ is ∞ -semiadditive if and only if this functor is an equivalence.

With this in place, we can define higher semiadditive K-theory by forcing the Segal condition.

Definition 15 (B.M.–Schlank). We define higher semiadditive K-theory as the composite

$$K^{[\infty]}: \text{Cat}_{p\text{-fin}}^{\text{st}} \simeq \text{CMon}_{\infty}(\text{Cat}_{p\text{-fin}}^{\text{st}}) \xrightarrow{K} \text{Fun}(\text{Span}(\mathcal{S}_{p\text{-fin}}), \text{Sp}_{(p)}) \xrightarrow{L^{\text{seg}}} \text{CMon}_{\infty}(\text{Sp}_{(p)}).$$

Notice that $|B^n C_p|$ being invertible is, of course, preserved under any ∞ -semiadditive functor, so we get a redshift upper bound for semiadditive K-theory.

Theorem 16. *Let $R \in \text{Alg}(\text{Sp}_{T(n)})$, then*

$$\text{ht}^{\oplus}(K^{[\infty]}(R)) \leq n+1.$$

If R has a height n root of unity (e.g. $\mathbb{S}_{T(n)}[\omega_{p^k}^{(n)}]$ or E_n) then

$$\text{ht}^{\oplus}(K^{[\infty]}(R)) = n+1.$$

Note that this construction made no reference to chromatic localization etc., but redshift comes entirely from the categorification. To close, I would like to connect this to chromatically localized K-theory. Since $T(i)$ -localization is an example of a left adjoint functor into an ∞ -semiadditive target, it canonically factors as follows

$$\text{Sp}_{(p)} \longrightarrow \text{CMon}_{\infty}(\text{Sp}_{(p)}) \xrightarrow[\leftarrow]{\dashv} \text{Sp}_{T(i)}$$

Moreover, the latter map is a (smashing!) localization, by a theorem of Carmeli–Schlank–Yanovski, so we get a comparison map $K^{[\infty]} \rightarrow L_{T(i)} K^{[\infty]}$. From this perspective, the content of the higher semiadditivity of $K_{T(n+1)}$ says that $L_{T(n+1)} K^{[\infty]}(\text{Dbl}(R)) \simeq K_{T(n+1)}(R)$. We conjecture that in fact the localization is not needed.

Conjecture 17 (B.M.–Schlank). *For $R \in \text{Alg}(\text{Sp}_{T(n)})$ there is an isomorphism*

$$K^{[\infty]}(R) \xrightarrow{\sim} K_{T(n+1)}(R).$$

We have proved this conjecture for the case going from height 0 to 1, that is if R is p -invertible, and the complete Johnson–Wilson spectrum $\widehat{E}(n)$ at any height. With Neugebauer, we are working on the case going from height 1 to 2, where we for example know this at the prime $p = 2$.

To phrase the conjecture in words, this would mean that $T(n+1)$ -localization of K -theory is completely characterized by being higher semiadditive. This, on the one hand, says that there is a non-chromatic interpretation for the chromatic localization, and on the other hand, that the lower chromatic heights are as far as possible from being higher semiadditive.