Delta Power Operations

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The goal in this talk is to explain the construction of the δ-operation of Section 4 of https://arxiv.org/abs/1811.02057, which is used in the proof of the main theorem explained in the previous talk. However, this talk is independent of many of the previous talks, and will only depend on a few basics things about 1-semiadditivity, so hopefully if you are a bit lost, you will still be able to follow this talk and see an example of how semiadditivity is utilized.

I will not follow the reference exactly, rather I will give a simpler presentation of the same objects (and some generalization).

1 Additive p-Derivations

We begin by defining and studying additive p-derivations on (ordinary) rings.

Definition 1. Let R be a commutative ring. An *additive p-derivation* on R is a function of sets $\delta\colon R \to R$ satisfying

1. (normalization) $\delta(0) = \delta(1) = 0$.
2. (additivity) $\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$ (note that the last term is in fact a polynomial with integer coefficients, involving no division by p.)

Remark 2. The axioms guarantee that $\varphi_\delta(x) = x^p + p\delta(x)$ is an *additive* lift of Frobenius, so the additivity axiom can be understood as the correction necessary to make φ_δ additive. (When R is p-torsion free, δ and φ_δ determine each other, but this is not true generally.)

Remark 3. You may be familiar with the notion of p-derivation, which imposes a further multiplicative condition, which also makes φ_δ above a ring homomorphism. We will not impose this axiom.

Example 4. The Fermat quotient on $\mathbb{Z}$ given by $\tilde{\delta}(x) := \frac{x - x^p}{p}$. Recall that Fermat's little theorem indeed says that $x^p = x \mod p$, so that this function is well defined. In fact, it is fairly straightforward to show that $\tilde{\delta}$ is the *unique* p-derivation on $\mathbb{Z}$. The crucial property of $\tilde{\delta}$ is that it *decreases the p-adic valuation by* 1 (when it is >0). To see this, note that $\nu_p\left(x^p\right)=p\nu_p\left(x\right)>\nu_p\left(x\right)$, so that $\nu_p\left(\frac{x}{p}-\frac{x^p}{p}\right)=\nu_p\left(\frac{x}{p}\right)=\nu_p\left(x\right)-1$. Furthermore, $\tilde{\delta}$ can be extended to $\mathbb{Q}$.

Lemma 5. *For $x\in R$ and $n\in\mathbb{N}$ we have $\delta\left(nx\right)=n\delta\left(x\right)+\tilde{\delta}\left(n\right)x^p$.*

Proof. Straightforward induction using the additivity axiom. □

Note that the mere existence of a p-derivation on R is a very restrictive property, as demonstrated by the following

Proposition 6. *Assume that R is p-local, and admits some additive p-derivation δ. Then, every torsion element is also nilpotent. In particular, if $R\otimes\mathbb{Q}=0$ then $R=0$.*

Proof. Let x be a torsion element, since R is p-local we know that $p^mx=0$ for some m. Now, applying δ we get from the previous lemma $0=\delta\left(0\right)=\delta\left(p^mx\right)=p^m\delta\left(x\right)+\tilde{\delta}\left(p^m\right)x^p$. Multiplying by x, since $p^mx=0$ we get $\tilde{\delta}\left(p^m\right)x^{p+1}=$, and as $\tilde{\delta}\left(p^m\right)$ has p-adic valuation $m-1$ so we get $p^{m-1}x^{p+1}$. Iterating this m times, we get $x^{(p+1)^m}=0$.

For the last part, if $R\otimes\mathbb{Q}=0$ then 1 is torsion, so in particular it is nilpotent, meaning that $1=1^n=0$, i.e. $R=0$. □

Example 7. As an anti-example, $\mathbb{Z}/p^m$ has no additive p-derivations.

2 Construction of the δ Power Operation

Our next goal is to construct such a δ on certain rings coming from homotopy, generalizing from the rational case (height 0) where we have $\tilde{\delta}$, to higher heights. More specifically, let $\mathcal{C}$ be a symmetric monoidal 1-semiadditive p-local stable presentable ∞-category, like $\mathrm{Sp}_{\mathrm{K}(n)}$ and $\mathrm{Sp}_{\mathrm{T}(n)}$. Let $X\in\mathrm{CAlg}\left(\mathcal{C}\right)$, then $R=\pi_0X=\pi_0\hom\left(\mathbf{1}_{\mathcal{C}},X\right)$ has a commutative ring structure. We will endow it with an additive p-derivation $\delta\colon R\to R$.

Here's the plan - first we will construct a family of power operations $\alpha_{G,A}\colon R\to R$ using 1-semiadditivity and the multiplicative structure of X (none of which is δ). Then, we will explain how in the case $\mathcal{C}=\mathrm{Sp}_{\mathbb{Q}}$ these operations reproduce $\frac{x}{p}$ and $\frac{x^p}{p}$, from which we can construct the Fermat quotient $\tilde{\delta}\left(x\right)=\frac{x-x^p}{p}$. Following these steps in the general case will give us δ.

Let A be a finite set with a G-action for some finite group G, we will construct an operation $\alpha_{G,A}\colon R\to R$. Let $x\in R$, and consider it as an element $x\in\hom\left(\mathbf{1}_{\mathcal{C}},X\right)$. We claim that $x^{|A|} \in \hom(\mathbf{1}_{\mathcal{C}}, X)$ has an $\mathrm{Aut}(A)$-action, coming from the commutativity of X.

$$\begin{tikzcd}
\mathbf{1}_{\mathcal{C}} \arrow[r] \arrow[d] & X^{\otimes A} \arrow[r] \arrow[d] & X \\
\mathbf{1}_{\mathcal{C}}/\mathrm{Aut}(A) \arrow[r] & X^{\otimes A}/\mathrm{Aut}(A) \arrow[ur] &
\end{tikzcd}$$

The top composition is by definition $x^{|A|}$. Since the multiplication on X is commutative, and we are taking the product of an element with itself, we have a factorization through the $\mathrm{Aut}(A)$-orbits. This gives us a map $\mathrm{BAut}(A) \to \hom(\mathbf{1}_{\mathcal{C}}, X)$ whose value at a point is by definition $x^{|A|}$, i.e. endows it with an $\mathrm{Aut}(A)$-action. Now, since A has a G-action, we can pre-compose to get an action of G, that is $\mathrm{B}G \to \mathrm{BAut}(A) \to X$.

Definition 8. We define $\alpha_{G,A} \colon R \to R$ by $\alpha_{G,A}(x) = \int_{\mathrm{B}G} x^{|A|}$.

Remark 9. The construction did not use the whole $\mathbb{E}_\infty$-structure but only the power operations, so it suffices to assume that X is an H_∞-algebra.

We now consider this construction in the rational case, where it simplifies significantly.

Proposition 10. *Let $\mathcal{C} = \mathrm{Sp}_{\mathbb{Q}}$, then $\alpha_{G,A}(x) = \int_{\mathrm{B}G} x^{|A|} = \frac{x^{|A|}}{|G|}$.*

Proof. We have essentially seen this, i.e. that generally in the rational case, $\int_{\mathrm{B}G} y = \frac{y}{|G|}$, let me repeat the argument. First note that the G-action is trivial, because there are no non-trivial maps $\mathrm{B}G \to X$ from B of a finite groups to a rational spectrum (this is essentially the fact that the rational cohomology of finite groups vanishes). Second, the norm map of a constant local system in the rational case is multiplication by $|G|$, whose inverse is $\frac{1}{|G|}$, which finishes the argument. $\square$

Example 11. For $A = *$ with the trivial $G = C_p$ action, $\alpha_{C_p,*}(x) = \int_{\mathrm{B}C_p} x = \frac{x}{p}$.

Example 12. For $A = C_p$ with the regular $G = C_p$ action, $\alpha_{C_p,C_p}(x) = \int_{\mathrm{B}C_p} x^p = \frac{x^p}{p}$.

Corollary 13. *For $\mathcal{C} = \mathrm{Sp}_{\mathbb{Q}}$ we have $\tilde{\delta}(x) = \frac{x - x^p}{p} = \alpha_{C_p,*}(x) - \alpha_{C_p,C_p}(x)$.*

This can now be made into a definition.

Definition 14. Let $\mathcal{C}$ be as above and $X \in \mathrm{CAlg}(\mathcal{C})$, and consider $R = \pi_0 X$. We define $\delta \colon R \to R$ by $\delta(x) := \alpha_{C_p,*}(x) - \alpha_{C_p,C_p}(x) = \int_{\mathrm{B}C_p} x - \int_{\mathrm{B}C_p} x^p$.

Theorem 15. *δ is an additive p-derivation on R.*

Remark 16. Note that the C_p action on C_p did not matter in the rational situation, but it will be used to prove that δ is an additive p-derivation in general.

We shall return to the proof after we draw a few corollaries.

Corollary 17. *Every torsion element of R is also nilpotent. In particular, if $R\otimes\mathbb{Q}=0$ then $X=0$.*

Proof. Follows immediately from the corresponding fact for rings, and the fact that if $\pi_0 X=0$ then the unit of X is 0 so that $X=0$. $\square$

3 May's Conjecture

May's conjecture was proven by Mathew-Naumann-Noel in 2014. We will demonstrate an alternative proof, which is immediate using the above considerations (together with the Nilpotence theorem). I find this example interesting, because it considers ring spectra not in the 1-semiadditive situation, but semiadditivity consideration can still solve it.

Remark 18. We say that a ring spectrum E *detects nilpotents* if for any ring spectrum X and $x\in\pi_* X$ whose image in $\pi_*(X\otimes E)$ is nilpotent, x itself is nilpotent. This is equivalent to having $X\otimes E=0 \implies X=0$, by applying it to $x^{-1}X$.

Corollary 19. *Let $X\in\mathrm{CAlg}(\mathrm{Sp})$ such that $X\otimes\mathbb{Q}=0$, then $L_{\mathrm{T}(n)}X=0$ for any $0\le n<\infty$, which also implies $L_{\mathrm{K}(n)}X=0$.*

Proof. We know that the unit $1\in X$ is torsion, so the same holds for $L_{\mathrm{T}(n)}X\in\mathrm{CAlg}\left(\mathrm{Sp}_{\mathrm{T}(n)}\right)$ which then satisfies the previous corollary, so that $L_{\mathrm{T}(n)}X=0$. $\square$

Corollary 20. *Let $X\in\mathrm{CAlg}(\mathrm{Sp})$ such that $X\otimes\mathbb{Z}=0$, then $X=0$.*

Proof. By assumption $X\otimes\mathbb{Z}=0$ so that also $X\otimes\mathbb{Q}=(X\otimes\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}=0$ and similarly $X\otimes\mathbb{F}_p=0$. By the previous part, we get that $L_{\mathrm{K}(n)}X=0$, i.e. $\mathrm{K}(n)\otimes X=0$, for every finite n. We conclude the proof by the Nilpotence Theorem. $\square$

Remark 21. As before, don't need the $\mathbb{E}_\infty$-structure, and H_∞-algebra suffices.

4 Proof of the Main Theorem

Theorem 22. $\delta(x)=\int_{\mathrm{B}C_p}x-\int_{\mathrm{B}C_p}x^p$ *is an additive p-derivation on R.*

Proof. For the normalization part, first it is clear that $\delta(0)=0$. For $x=1$, one can easily show that the C_p action on $1^p=1$ is trivial, so that $\delta(1)=\int_{\mathrm{B}C_p}1-\int_{\mathrm{B}C_p}1=0$.

We need to show that additivity condition $\delta(x+y)=\delta(x)+\delta(y)+\frac{x^p+y^p-(x+y)^p}{p}$. Clearly $\int_{\mathrm{B}C_p}(x+y)=\int_{\mathrm{B}C_p}x+\int_{\mathrm{B}C_p}y$, so we shall show that $\int_{\mathrm{B}C_p}(x+y)^p=\int_{\mathrm{B}C_p}x^p+\int_{\mathrm{B}C_p}y^p+\frac{(x+y)^p-x^p-y^p}{p}$ (note that the numerator is reversed, compatibly with the minus sign).

The idea is to open the parentheses of $(x+y)^p$, and combine things by their C_p-orbit. Let S be the set of all words of length p in 2 letters which are not all the same, then

$$(x+y)^p = x^p + y^p + \sum_{w\in S} w(x,y) = x^p + y^p + \sum_{[w]\in S/C_p} \sum_{g\in C_p} gw(x,y).$$

Each orbit $[w] \in S/C_p$ yields a term $\sum_{g\in C_p} gw(x,y)$. Note that this the sum of $w(x,y)$ along the C_p-orbit, namely if we let $p\colon * \to \mathrm{B}C_p$ be the map choosing the point, whose fibers are C_p, then $\int_p w(x,y) = \sum_{g\in C_p} gw(x,y)$. Then, we wish to integrate this using $\int_{\mathrm{B}C_p}$, and recall this is just $\int_q$ for $q\colon \mathrm{B}C_p \to *$. Note that $qp\colon * \to *$ is the identity, which gives us

$$\int_{\mathrm{B}C_p} \sum_{g\in C_p} gw(x,y) = \int_q \int_p w(x,y) = \int_{qp} w(x,y) = w(x,y).$$

Note that commutativity, without the C_p-action, we have $w(x,y) = x^{w_x} y^{w_y} = \frac{\sum_{g\in C_p} x^{w_x} y^{w_y}}{p}$, so we just get the binomial terms

$$\begin{aligned}
\int_{\mathrm{B}C_p} \sum_{w\in S} w(x,y) &= \int_{\mathrm{B}C_p} \sum_{[w]\in S/C_p} \sum_{g\in C_p} gw(x,y) \\
&= \sum_{[w]\in S/C_p} \int_{\mathrm{B}C_p} \sum_{g\in C_p} gw(x,y) \\
&= \frac{\sum_{[w]\in S/C_p} \sum_{g\in C_p} x^{w_x} y^{w_y}}{p} \\
&= \frac{\sum_{w\in S} x^{w_x} y^{w_y}}{p} \\
&= \frac{(x+y)^p - x^p - y^p}{p}.
\end{aligned}$$

□