SAG Seminar – Lubin-Tate Deformation Theory

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Some useful references for this talk are as follows. The classical paper is [1]. A good reference with a lot of details for homotopists is [3]. For a short overview, which I find a little difficult to follow, but takes a simpler route to the theorem is in [2, mostly lecture 21]. A more modern approach, which gives a very nice overview, but not very detailed is [4].

This lecture will consider something fairly classical, the theory of deformations of formal group laws. This in essence to (non-spectral) algebraic geometry and number theory, and doesn't use any homotopy or higher category theory (actually a little bit of 2-categories in the way we phrase things). This of course has connections to homotopy theory, via chromatic homotopy theory, but we will not take about this at all in this lecture. The goal of the first and second parts of this talk, which are the most important parts, is to give an exposition of the constructions and the main theorem from a modern point of view. In the third part we will take the classical point of view on this theory. Then in the fourth part we will consider the functoriality of this construction, again from the more modern perspective. Lastly, if we have time, we will say something on the Morava stabilizer group. Hopefully this lecture will also be a good supplement to standard material on this theory.

1 Formal Group Laws & Height

We first recall our definition of a formal group law. We will take the coordinate-free approach we have already seen in this course, which has the advantage of being conceptually clearer. However, actually the main statement will be completely independent of the specific model we choose for formal group laws. Furthermore, we will reinterpret everything later in the classical way it was done, which may help to make things more elementary, and will also give us some insights.

Let R be a ring. For a finitely generated projective module M over R, we defined $\mathrm{Spf}\left(R\left[\left[M\right]\right]\right) : \mathrm{CAlg}_R \to \mathrm{Set}$. A *formal scheme* is a functor $\mathrm{CAlg}_R \to \mathrm{Set}$ that is isomorphic to some $\mathrm{Spf}\left(R\left[\left[M\right]\right]\right)$, but without the data of the isomorphism.

A *formal group* over R is a lift of such a functor to Grp, i.e. it is a functor $\mathbb{G} : \mathrm{CAlg}_R \to \mathrm{Grp}$, such that the composition with the forgetful $\underline{\mathbb{G}} : \mathrm{CAlg}_R \to \mathrm{Grp} \to \mathrm{Set}$ is isomorphic to $\mathrm{Spf}\left(R\left[\left[M\right]\right]\right)$ for some M. Note that given a map $\alpha : R \to S$, which should be though of as $\alpha : \mathrm{Spec}\, S \to \mathrm{Spec}\, R$, we can define $\alpha^* \mathbb{G} : \mathrm{CAlg}_S \to \mathrm{CAlg}_R \to \mathrm{Grp}$. We also have a definition of the tangent space, $T\mathbb{G} = \mathbb{G}\left(R\left[x\right]/x^2\right)$, that can further be endowed with the structure of a module over R, and we have seen that $T^*\mathbb{G} \cong M$ (warning: this doesn't that we get an isomorphism from $\mathbb{G}$ to $\mathrm{Spf}\left(R\left[\left[M\right]\right]\right)$).

Consider the special case $M = R\left\{x\right\}$, we get $\mathrm{Spf}\left(R\left[\left[x\right]\right]\right)(A) = \mathrm{colim}\,\hom\left(R\left[x\right]/x^n, A\right) = \hom_{\mathrm{cont}}\left(R\left[\left[x\right]\right], A\right) = \sqrt{A}$.

Definition 1. A formal group $\mathbb{G}$ together with an isomorphism of the cotangent space $T^*\mathbb{G} \xrightarrow{\sim} R\left\{x\right\}$ is called a (1-dimensional commutative) *formal group law* and the choice of the isomorphism is called a *coordinate*. Naturally, an isomorphism of formal group laws is required to commute with the coordinate, usually called a strict-isomorphism. The groupoid of formal group laws over R together with strict-isomorphisms is denoted by $\mathrm{FGL}(R)$.

Example 2. The *additive formal group law* is defined by $\hat{\mathbb{G}}_a(A) = \sqrt{A}$ with the usual addition, $x, y \mapsto x + y$.

Example 3. The *multiplicative formal group law* is defined by $\hat{\mathbb{G}}_m(A) = \sqrt{A}$ with the operation given by the identification of $\sqrt{A}$ with $1 + \sqrt{A}$ and using the multiplicative structure, $x, y \mapsto (1+x)(1+y) - 1 = x + y + xy$.

Let's recall some results from the previous time:

Theorem 4 (/definition)**.** *Let $\mathbb{G}$ be a formal group law over R,*

1. *if R is a $\mathbb{Q}$-algebra, $\mathbb{G} \cong \hat{\mathbb{G}}_a$ ($\mathbb{G}[p]$ is $1 = p^0$ dimensional), and we define* $\mathrm{ht}(\mathbb{G}) = 0$*,*
2. *if R is a field of characteristic p, either*
   (a) *the p-torsion $\mathbb{G}[p]$ is an algebra of dimension p^n for some $1 \leq n < \infty$ and we define* $\mathrm{ht}(\mathbb{G}) = n$*,*
   (b) *or $\mathbb{G} \cong \hat{\mathbb{G}}_a$ ($\mathbb{G}[p]$ is infinite dimensional), and we define* $\mathrm{ht}(\mathbb{G}) = \infty$.
3. *for an arbitrary ring, we define* $\mathrm{ht}(\mathbb{G}) : \mathrm{Spec}\, R \to \mathbb{Z}_{\geq 0}$ *by* $\mathrm{ht}(\mathbb{G})(\mathfrak{p}) = \mathrm{ht}\left(\mathbb{G}\mid_{\kappa(\mathfrak{p})}\right)$*, where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}$ is the residue field.*

2 Lubin-Tate Deformation Theory

Let's fix a perfect field of characteristic p, k (to be concrete think of $\mathbb{F}_p$ or $\overline{\mathbb{F}}_p$), and a formal group law over it Γ of some finite height $n = \mathrm{ht}(\Gamma) < \infty$. One may wonder, which formal group laws are "close" to Γ, in some sense. The formalization of this is given by the Lubin-Tate deformation theory. This is a moduli problem given by a functor $\mathrm{Def} : \mathrm{CompRings} \to \mathrm{Grpd}$. Let $(R, \mathfrak{m})$ be a complete local ring, and denote by $\pi : R \to R/\mathfrak{m}$ the projection. Again, promoting the view that we should take Spec on all fields and rings, this data should be though of as a thickening of the point $\mathrm{Spec}\, R/\mathfrak{m}$.

Then, $\mathrm{Def}\,(R, \mathfrak{m})$ is abstractly given by the pullback:

$$\begin{array}{ccc} \mathrm{Def}\,(R,\mathfrak{m}) & \longrightarrow & \mathrm{FGL}\,(R) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mathbb{G}\mapsto\pi^*\mathbb{G}} \\ \{i : k \to R/\mathfrak{m}\} & \xrightarrow{i\mapsto i^*\Gamma} & \mathrm{FGL}\,(R/\mathfrak{m}) \end{array}$$

where the bottom left is the discrete groupoid of homomorphisms of fields.

Concretely, a *deformation* of Γ to $(R, \mathfrak{m})$ is $(\mathbb{G}, i, \tau)$, where $\mathbb{G}$ is a formal group law over R, $i : k \to R/\mathfrak{m}$, and $\tau : i^*\Gamma \xrightarrow{\sim} \pi^*\mathbb{G}$ is a strict-isomorphism. Concisely, this is a lift of Γ, base changed to $R/\mathfrak{m}$. We will always denote this by:

$$\begin{array}{c} \pi^*\mathbb{G} \\ {\scriptstyle\tau}\big\uparrow \\ i^*\Gamma \end{array}$$

An isomorphism of such triples $(\mathbb{G}_1, i_1, \tau_1) \to (\mathbb{G}_2, i_2, \tau_2)$ is usually called $\star$-isomorphism, it exists only when $i_1 = i = i_2$, and then it is a strict-isomorphism $g : \mathbb{G}_1 \to \mathbb{G}_2$ compatible with τ_1, τ_2 in the sense that the following diagram commutes:

$$\begin{array}{ccc} \pi^*\mathbb{G}_1 & \xrightarrow{\pi^* g} & \pi^*\mathbb{G}_2 \\ {\scriptstyle\tau_1}\big\uparrow & & {\scriptstyle\tau_2}\big\uparrow \\ i^*\Gamma & \xrightarrow{\mathrm{id}} & i^*\Gamma \end{array}$$

We note that $i^*\Gamma$ has the same height as Γ, and taking a lift from $R/\mathfrak{m}$ to R can only decrease the height so $\mathbb{G}$ satisfies $0 \leq \mathrm{ht}\,(\mathbb{G})\,(-) \leq n$ (this will be clearer in the classical point of view later).

Theorem 5. *The functor* $\mathrm{Def} : \mathrm{CompRings} \to \mathrm{Grpd}$*,*

1. *lands in discrete groupoids (i.e. there are either* 0 *or* 1 *morphisms between objects, that is, sets),*

2. *is* corepresented*, that is there is a complete local ring* $(R_U, \mathfrak{m}_U)$ *such that for each* $(R, \mathfrak{m})$ *we have* $\hom\left((R_U, \mathfrak{m}_U), (R, \mathfrak{m})\right) \xrightarrow{\sim} \mathrm{Def}\,(R, \mathfrak{m})$*.*

To be more specific, the ring R_U can be chosen to be $Wk\,[[u_1, \ldots, u_{n-1}]]$. Here Wk is the Witt ring of k, which is a complete local ring with maximal ideal

(p) and residue field k, and is some sort of a formal neighborhood of k. As an example, for $k = \mathbb{F}_p$ we get $Wk = \mathbb{Z}_p$, and indeed $\mathbb{Z}_p/p = \mathbb{F}_p$. It then makes sense to denote $u_0 = p$, and the maximal ideal of R_U is $\mathfrak{m}_U = (u_0, u_1, \ldots, u_{n-1})$.

Further (by taking $\mathrm{id} : R_U \to R_U$), there is a universal deformation $(\mathbb{G}_U, \mathrm{id}, \tau_U)$, i.e. there is a formal group law $\mathbb{G}_U$ with $\tau_U : \Gamma \xrightarrow{\sim} \pi^* \mathbb{G}_U$, such that for any complete local ring the map $\hom\left((R_U, \mathfrak{m}_U), (R, \mathfrak{m})\right) \to \mathrm{Def}\,(R, \mathfrak{m})$ given by sending $\varphi : R_U \to R$ to $\left(\varphi^* \mathbb{G}_U, \varphi/\mathfrak{m}, (\varphi/\mathfrak{m})^* \Gamma \xrightarrow{(\varphi/\mathfrak{m})^* \tau_U} (\varphi/\mathfrak{m})^* \pi^* \mathbb{G}_U = \pi^* \varphi^* \mathbb{G}_U\right)$ is a bijection (the target is a discrete groupoid, so we mean equivalence of groupoids).

$$\begin{array}{ccc}
\pi^* \mathbb{G}_U & \rightsquigarrow & \pi^* \varphi^* \mathbb{G}_U = (\varphi/\mathfrak{m})^* \pi^* \mathbb{G}_U \\
{\scriptstyle \tau_U} \uparrow & & \uparrow {\scriptstyle (\varphi/\mathfrak{m})^* \tau_U} \\
\Gamma & \rightsquigarrow & (\varphi/\mathfrak{m})^* \Gamma
\end{array}$$

Recall that the height of any deformation, and in particular that of $\mathbb{G}_U$, satisfies $0 \leq \mathrm{ht}\,(\mathbb{G})\,(-) \leq n$. The theorem actually tells us that $\mathbb{G}_U$ attains all those heights, i.e. it interpolates between 0 and n. For each t we have the prime ideal $\mathfrak{p}_t = (u_0, u_1, \ldots, u_{t-1})$. The theorem shows that $\mathrm{ht}\,(\mathbb{G}_U)\,(\mathfrak{p}_t) = t$. In particular, over the generic fiber $Wk\left[p^{-1}, u_1^{\pm 1}, \ldots, u_{n-1}^{\pm 1}\right]$ the height is 0, and over the fiber k the height is (of course) n.

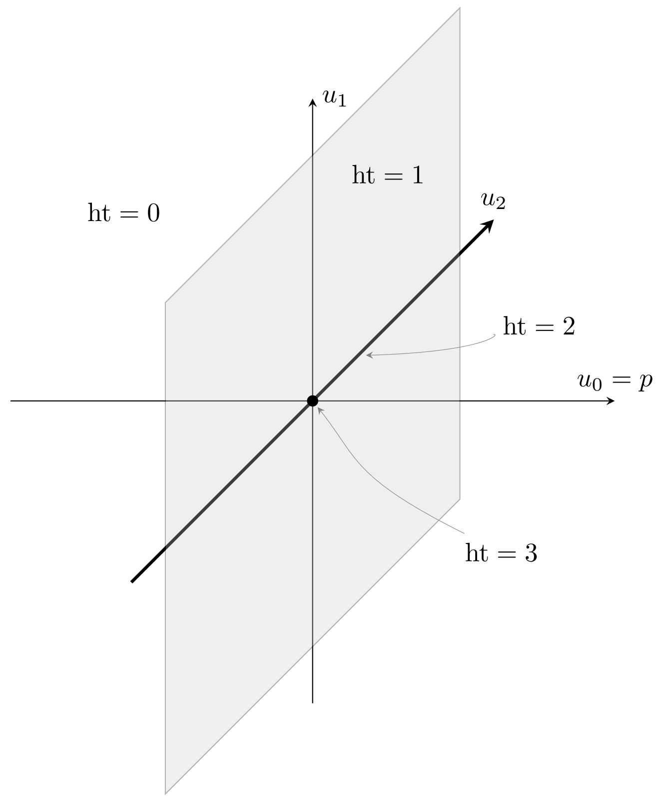

Figure 1: Illustration of $\mathrm{ht}(\mathbb{G}_U) : \operatorname{Spec} R_U \to \mathbb{Z}_{\geq 0}$ for the case $n = 3$. The interpretation of the axes is that $u_i = 0$ at the 0 of the axis, and u_i is invertible everywhere else.

3 Classical Point of View

Let's reinterpret this classically. First of all, recall that given a formal group law $\mathbb{G}$ over R, we can choose an isomorphism $\mathbb{G} \xrightarrow{\sim} \operatorname{Spf}(R[[x]])$ (although it is not unique). The data of the multiplication is (by Yoneda) equivalent to a map $\operatorname{Spf}(R[[y]]) \times \operatorname{Spf}(R[[z]]) \to \operatorname{Spf}(R[[x]])$, equivalently a map $R[[x]] \to R[[y]] \hat{\otimes} R[[z]] = R[[y,z]]$, which is determined by the image of x, which we denote by $F_{\mathbb{G}}(y,z)$ or $y +_{\mathbb{G}} z$. This "addition" operation has 0 as a neutral element, it is associative and commutative, and any such formal power series $R[[y,z]]$ determines a formal group law. A strict-isomorphism $g : \mathbb{G}_1 \to \mathbb{G}_2$ is given by a power series $H_g(x)$, which is on the opposite direction on the algebraic side, so it satisfies $H_g(y) +_{\mathbb{G}_1} H_g(z) = H_g(y +_{\mathbb{G}_2} z)$, and $H_g(x) = x \mod x^2$ (because it commutes with the coordinate). Furthermore $\alpha : R \to S$ sends $F_{\mathbb{G}}$ to $\alpha^* F_{\mathbb{G}} = F_{\alpha^* \mathbb{G}}$ by applying α to the coefficients in the formal power series.

Well, then the groupoid $\mathrm{FGL}(R)$ of formal group laws over R is equivalent to the groupoid of such power series $F(y,z)$ and such $H(x)$ as morphisms. So we can define $\mathrm{Def}(R,\mathfrak{m})$ in these terms. Actually we can rigidify as follows: a deformation of $F_\Gamma \in k[[y,z]]$ to $(R,\mathfrak{m})$ is a pair (F,i) such that $i^*F_\Gamma = \pi^*F$ (rather than isomorphic), i.e. $F = i^*F_\Gamma \mod \mathfrak{m}$. A $\star$-isomorphism $(F_1,i_1) \to (F_2,i_2)$ again exists only when $i_1 = i = i_2$, and it is a strict-isomorphism $H \in R[[x]]$, such that $H(x) = x \mod \mathfrak{m}$ (i.e. the identity when base changed using π^*).

Example 6. By definition, for the additive formal group law $F_{\hat{\mathbb{G}}_a}(x,y) = x+y$, and for the multiplicative formal group law $F_{\hat{\mathbb{G}}_m}(x,y) = (1+x)(1+y) - 1 = x+y+xy$.

Recall that the *height* of a formal group law was defined to be n where the dimension of $\mathbb{G}[p]$ is p^n. This can also be seen as follows. Given the choice of the formal power series as above, we can form the p-series $[p]_{\mathbb{G}}(x) = x +_{\mathbb{G}} \cdots +_{\mathbb{G}} x$. Since to first order $x +_{\mathbb{G}} y = x + y$, to first order $[p]_{\mathbb{G}}(x) = px$. So over a $\mathbb{Q}$-algebra the coefficient of $x = x^{p^0}$ is p which is invertible, and the height is 0. Over a field (actually ring) of characteristic p, the p-series is either all 0 so the height is ∞, or, as it turns out, it is power series in x^{p^i} with non-vanishing coefficient for x^{p^i} for some i, i.e. it is of the form $[p]_{\mathbb{G}}(x) = v_i x^{p^i} + \cdots$ with $v_i \neq 0$. This i is the height.

Example 7. For the additive formal group law, we get $[p]_{\hat{\mathbb{G}}_a}(x) = px$. Therefore over a $\mathbb{Q}$-algebra we easily see that the height is 0. Over a field of characteristic p, we see that $[p]_{\hat{\mathbb{G}}_a} = 0$ so the height is ∞.

Example 8. For the multiplicative formal group law, we get $[p]_{\hat{\mathbb{G}}_m}(x) = (1+x)^p - 1$. Again, over a $\mathbb{Q}$-algebra this starts with px. Over a field of characteristic p, we see that $[p]_{\hat{\mathbb{G}}_a} = (1+x)^p - 1 = 1 + x^p - 1 = x^p$ so the height is 1.

In our case we start with a formal group law Γ of height $1 \leq n < \infty$ over k, so the p-series is of the form $[p]_\Gamma = \overline{a}x^{p^n} + \cdots$. And we wish to construct a corresponding formal group law $\mathbb{G}_U$ over $R_U = Wk[[u_1, \ldots, u_{n-1}]]$. We shall give a sketch of the construction following Lurie [2, lectures 13, 21], who takes a different approach then what is usually done.

Recall that there is a universal formal group law F_{univ} over the Lazard ring L. This is given by taking $L = \mathbb{Z}[a_{ij}] / \sim$ and $F_{\mathrm{univ}}(y,z) = \sum a_{ij} y^i z^j$, where $\sim$ are all the relations dictated by the axioms of a formal group law. Then a homomorphism $L \to R$ is the same data as a formal group law on R by $\varphi \mapsto \varphi^* F_{\mathrm{univ}}$. Lazard theorem states that $L \cong \mathbb{Z}[t_1, t_2, \ldots]$ (non-canonically). Now, assuming R is p-local, i.e. a $\mathbb{Z}_{(p)}$-algebra, maps $L \to R$ are the same as maps $L_{(p)} \to R$. We can write the p-series of the universal formal group law And it turns out that after the p-localization, $[p]_{F_{\mathrm{univ}}}(x)$, and denote the coefficient

v_i of x^{p^i} (which is polynomial in the t_j). It turns out that after p-localization we can choose the $t_{p^i-1} = v_i$ in $L_{(p)} \cong \mathbb{Z}_{(p)}[t_1, t_2, \dots]$.

Since Γ over k has height n, it is given by a map $L_{(p)} \to k$ where $v_i \mapsto 0$ for $i < n$ and $v_n \mapsto \overline{a}$ (since $[p]_\Gamma = \overline{a}x^{p^n} + \cdots$). Now choose any lift $L_{(p)} \to Wk[[u_1, \dots, u_{n-1}]]$ such that $v_i \mapsto u_i$ for $i < n$ (automatically $v_0 = p = u_0$) and $v_n \mapsto a$ some lift of $\overline{a}$. The claim is that any such choice corresponds to a formal group law which gives a universal deformation, we will not prove this.

However, we can easily verify that $\mathrm{ht}(\mathbb{G}_U)(\mathfrak{p}_t) = t$ for $\mathfrak{p}_t = (u_0, u_1, \dots, u_{t-1})$. Indeed, for $t = 0$ we have $\mathfrak{p}_0 = 0$, we get a $\mathbb{Q}$-algebra (since we invert $u_0 = p$), and the height is 0. For $1 \le t \le n$, the field $\kappa(\mathfrak{p}_t)$ is of characteristic p since we take the quotient by $u_0 = p$. So the p-series is a power series in x^{p^i} for some i. We fixed $v_i \mapsto u_i$, so the coefficient of x^{p^i} is u_i, which is 0 for $i < t$, and non-zero for $i = t$, i.e. it is $u_t x^{p^t} + \cdots + u_{n-1}x^{p^{(n-1)}} + \cdots + ax^{p^n} + \cdots$, so we get $\mathrm{ht}(\mathbb{G}_U)(\mathfrak{p}_t) = t$.

Example 9. Take the multiplicative formal group law $\hat{\mathbb{G}}_m$ over $\mathbb{F}_p$. We saw that it is of height $n = \mathrm{ht}\left(\hat{\mathbb{G}}_m\right) = 1$. Therefore $R_U = W\mathbb{F}_p = \mathbb{Z}_p$. In this case $\mathbb{G}_U = \hat{\mathbb{G}}_m$ over $\mathbb{Z}_p$ is a universal deformation (we have essentially seen the height computations before).

To wrap up, given a formal group law Γ of height n over k, there is a universal deformation $\mathbb{G}_U$ over R_U, that has height between 0 and n (we recall that these were all possible heights for a deformation). The meaning of this is that the infinitesimal neighborhood of a formal group law of height n, contains heights 0 to n, and the height there is generically 0 (and most specially n).

4 Functoriality

The category of formal group laws (in characteristic p, without reference to a ring), is denoted by FGL_p. Its objects are a pair (k, Γ) of a perfect field of characteristic p and a formal group law Γ over it. Again, we should think of $\mathrm{Spec}\, k$ rather than k, then a morphism $(k_1, \Gamma_1) \to (k_2, \Gamma_2)$ is a homomorphism $\overline{\varphi} : k_2 \to k_1$ and a strict-isomorphism $\overline{g} : \overline{\varphi}^*\Gamma_2 \to \Gamma_1$, in particular we note that a morphism exists only if Γ_1 and Γ_2 have the same height (as $\overline{\varphi}^*$ doesn't change the height). Similarly we denote by FGL_0 the category of pairs $((R, \mathfrak{m}), \mathbb{G})$ where $(R/\mathfrak{m}, \pi^*\mathbb{G}) \in \mathrm{FGL}_p$, and morphisms are similarly $\varphi : R_2 \to R_1$ and a strict-isomorphism $g : \varphi^*\mathbb{G}_2 \to \mathbb{G}_1$.

What we have seen up until now is an assignment $\mathrm{FGL}_p \to \mathrm{FGL}_0$, given by $(k, \Gamma) \mapsto (R_U, \mathbb{G}_U)$, which corepresents the moduli problem. We claim that this assignment is in fact functorial (actually we claim that the moduli problem itself is functorial, and this will follow).

Say we have a morphism $(k_1, \Gamma_1) \to (k_2, \Gamma_2)$, we build a natural transformation between the moduli problems $\mathrm{Def}_{k_1,\Gamma_1} \Rightarrow \mathrm{Def}_{k_2,\Gamma_2}$. Let $(R, \mathfrak{m})$ be a complete local ring. For a deformation $(\mathbb{G}, i, \tau)$ of Γ to R, i.e. $i : k_1 \to R/\mathfrak{m}$ and $\tau : i^*\Gamma_1 \xrightarrow{\sim} \pi^*\mathbb{G}$, we can construct the deformation $\left(\mathbb{G}, k_2 \xrightarrow{\overline{\varphi}} k_1 \xrightarrow{i} R/\mathfrak{m}, i^*\overline{\varphi}^*\Gamma_2 \xrightarrow{i^*\overline{g}} i^*\Gamma_1 \xrightarrow{\tau} \pi^*\mathbb{G}\right)$. Therefore we have a map $\mathrm{Def}_{k_1,\Gamma_1}(R, \mathfrak{m}) \to \mathrm{Def}_{k_2,\Gamma_2}(R, \mathfrak{m})$, and this assembles into a natural transformation $\mathrm{Def}_{k_1,\Gamma_1} \Rightarrow \mathrm{Def}_{k_2,\Gamma_2}$.

$$\begin{array}{ccc} & & \pi^*\mathbb{G} \\ & & \uparrow{\scriptstyle \tau} \\ i^*\overline{\varphi}^*\Gamma_2 & \xrightarrow{i^*\overline{g}} & i^*\Gamma_1 \end{array}$$

The functors are corepresented, so this is a (co-)Yoneda situation. Let's unravel what we actually get by taking the element corresponding to the identity on the first, i.e. the universal deformation $\left(\mathbb{G}_U^1, \mathrm{id}, \tau_U^1\right) \in \mathrm{Def}_{k_1,\Gamma_1}\left(R_U^1\right)$. By definition, it is mapped to $\left(\mathbb{G}_U^1, \overline{\varphi}, \tau_U^1 \circ \overline{g}\right) \in \mathrm{Def}_{k_2,\Gamma_2}\left(R_U^1\right)$. But this functor is corepresented as well, this time by R_U^2, so we have some $\varphi : R_U^2 \to R_U^1$, and a (unique) $\star$-isomorphism from $\left(\varphi^*\mathbb{G}_U^2, \varphi/\mathfrak{m}, (\varphi/\mathfrak{m})^* \tau_U^2\right)$ to $\left(\mathbb{G}_U^1, \overline{\varphi}, \tau_U^1 \circ \overline{g}\right)$. This means that $\varphi/\mathfrak{m} = \overline{\varphi}$, i.e. φ is a lift of $\overline{\varphi}$ which justifies the notation, and we have a strict-isomorphism $g : \varphi^*\mathbb{G}_U^2 \to \mathbb{G}_U^1$ such that the following diagram commutes:

$$\begin{array}{ccccc} \pi^*\varphi^*\mathbb{G}_U^2 & & \xrightarrow{\pi^* g} & & \pi^*\mathbb{G}_U^1 \\ \uparrow{\scriptstyle (\varphi/\mathfrak{m})^*\tau_U^2} & & & & \uparrow{\scriptstyle \tau_U^1} \\ (\varphi/\mathfrak{m})^*\Gamma_2 & \xrightarrow{\mathrm{id}} & \overline{\varphi}^*\Gamma_2 & \xrightarrow{\overline{g}} & \Gamma_1 \end{array}$$

i.e. g is a lift of $\overline{g}$ which again justifies the notation. To conclude, from $\overline{\varphi} : k_2 \to k_1$ and $\overline{g} : \overline{\varphi}^*\Gamma_2 \xrightarrow{\sim} \Gamma_1$ we constructed (uniquely) lifts $\varphi : R_U^2 \to R_U^1$ and $g : \varphi^*\mathbb{G}_U^2 \to \mathbb{G}_U^1$. This turns the construction $\mathrm{FGL}_p \to \mathrm{FGL}_0$ given by $(k, \Gamma) \mapsto (R_U, \mathbb{G}_U)$ into a functor.

5 Morava Stabilizer Group

Fixing $(k, \Gamma) \in \mathrm{FGL}_p$, we get by the functoriality a map $\mathrm{Aut}(k, \Gamma) \to \mathrm{Aut}(R_U, \mathbb{G}_U)$, i.e. the group $\mathrm{Aut}(k, \Gamma)$ the group acts on $R_U = Wk[[u_1, \ldots, u_{n-1}]]$, and twists $\mathbb{G}_U$ accordingly. This group is called the *n-th Extended Morava Stabilizer Group*, and usually denoted (unfortunately) by $\mathbb{G}_n = \mathrm{Aut}(k, \Gamma)$. It has a subgroup of those maps that act only on Γ (i.e. $\overline{\varphi} = \mathrm{id}$), which is called the *$n$-th Morava Stabilizer Group* and is usually denoted by S_n. The group $\mathbb{G}_n$ splits into a semi-direct product $\mathbb{G}_n = S_n \rtimes \mathrm{Gal}(k : \mathbb{F}_p)$.

We recall that over an algebraically closed field, such as $\overline{\mathbb{F}}_p$, there is a unique formal group law of height n up to strict-isomorphism, so in this case $\mathbb{G}_n$ and

S_n are independent of Γ, and usually people refer to these groups. Moreover, it turns out that past $\mathbb{F}_{p^n}$, the automorphisms of a formal group law of height n don't change, so one can take $k = \mathbb{F}_{p^n}$ instead.

References

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