A note on Mitchell's theorem

Shay Ben-Moshe

Elmanto–Nardin–Yang [ENY25] gave a short proof of Mitchell's theorem using several recent descent results. This note records the small observation that a slight modification of their 'warm-up' p-adic case can be adapted to prove the full theorem, using only classical results together with May's nilpotence conjecture, proved by Mathew–Naumann–Noel [MNN15] and later strengthened and reproved in [Hah16, CSY22]. Alternatively, as in [ENY25], one may replace the direct use of May's nilpotence by the finite-flat descent theorem of [CMNN20], which itself relies on May's nilpotence. We note that [CMNN24, Remark 4.4] gave a different short proof of Mitchell's theorem.

Theorem (Mitchell [Mit90])**.** $L_{\mathrm{T}(n)}\mathrm{K}(\mathbb{Z}) = 0$ for all primes p and $n \geq 2$.

Proof. $\underline{L_{\mathrm{T}(n)}\mathrm{K}(\overline{\mathbb{Q}}) = 0}$: We recall that by Suslin's rigidity theorem [Sus83, Sus84], the maps

$$\mathrm{K}(\overline{\mathbb{Q}}) \longrightarrow \mathrm{K}(\mathbb{C}) \longrightarrow \mathrm{ku}$$

are p-adic equivalences, and the claim follows since $L_{\mathrm{T}(n)}\mathrm{ku} = 0$.

$\underline{L_{\mathrm{T}(n)}\mathrm{K}(F) = 0 \text{ for some number field } F/\mathbb{Q}}$: Note that $\overline{\mathbb{Q}}$ is the filtered colimit over the number fields $F/\mathbb{Q}$. Since K preserves filtered colimits, and $L_{\mathrm{T}(n)}$ preserves colimits, we get

$$\operatorname*{colim}_{F/\mathbb{Q}} L_{\mathrm{T}(n)}\mathrm{K}(F) \xrightarrow{\sim} L_{\mathrm{T}(n)}\mathrm{K}(\overline{\mathbb{Q}}) = 0$$

Recall that a ring spectrum vanishes if and only if its unit vanishes. Since the colimit is along ring spectra maps, the unit of $L_{\mathrm{T}(n)}\mathrm{K}(F)$ vanishes for some $F/\mathbb{Q}$.

$\underline{L_{\mathrm{T}(n)}\mathrm{K}(\mathbb{Q}) = 0}$: The composition of the free and forgetful functors

$$\mathrm{Perf}(\mathbb{Q}) \xrightarrow{F\otimes_{\mathbb{Q}}-} \mathrm{Perf}(F) \xrightarrow{U} \mathrm{Perf}(\mathbb{Q})$$

induces multiplication by $\dim_{\mathbb{Q}}(F)$ on $\mathrm{K}(\mathbb{Q})$. Since $L_{\mathrm{T}(n)}\mathrm{K}(F) = 0$, we get that $\dim_{\mathbb{Q}}(F) \cdot 1 = 0$ in $L_{\mathrm{T}(n)}\mathrm{K}(\mathbb{Q})$, namely its unit is torsion. Since $L_{\mathrm{T}(n)}\mathrm{K}(\mathbb{Q})$ is an $\mathbb{F}_p$-acyclic commutative ring spectrum, we conclude using May's nilpotence [MNN15, Theorem B, see also Proposition 4.2] that its unit is nilpotent, hence $L_{\mathrm{T}(n)}\mathrm{K}(\mathbb{Q}) = 0$.

$\underline{L_{\mathrm{T}(n)}\mathrm{K}(\mathbb{Z}) = 0}$: Quillen's localization and devissage theorems [Qui73, §5] give an exact sequence

$$\bigoplus_{\ell \text{ prime}} \mathrm{K}(\mathbb{F}_\ell) \longrightarrow \mathrm{K}(\mathbb{Z}) \longrightarrow \mathrm{K}(\mathbb{Q}).$$

By Quillen's computation [Qui72] we have $L_{\mathrm{T}(n)}\mathrm{K}(\mathbb{F}_\ell) = 0$, which concludes the proof. □

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