Rotation Invariance Seminar: A Model for Bott Periodicity

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To situate this talk in the seminar, recall that we are interested in an action of S^1 on $\mathrm{Cat}_\infty^{\mathrm{st}}$, specified by the $\mathbb{E}_2$-map

$$\mathbb{Z} \xrightarrow{\sim} \Omega^2\mathrm{BU}(1) \longrightarrow \Omega^2(\mathbb{Z}\times\mathrm{BU}) \xrightarrow{\sim} \mathbb{Z}\times\mathrm{BU} \xrightarrow{J} \mathrm{Pic}(\mathbb{S}).$$

In this talk, we shall be interested in the third map – the Bott periodicity isomorphism. This is in fact an isomorphism of $\mathbb{E}_\infty$-spaces, but our goal is to understand it as an $\mathbb{E}_2$-map, via a topological model.

Let us begin with some opening remarks on the $\mathbb{E}_2$-structures on both sides, starting with $\mathbb{Z}\times\mathrm{BU}$. Consider the topologically enriched category $\mathrm{Vect}_{\mathbb{C}}$ of finite dimensional complex vector spaces, and where $\hom(V,W)$ is topologized via the isomorphism with $\mathbb{C}^{\dim V\times\dim W}$. Observe that the associated ∞-category $\mathrm{N}(\mathrm{Vect}_{\mathbb{C}})$ is trivial, since $\hom(V,W)$ is contractible (note that therefore in this talk it is important to distinguish between topological constructions and their homotopical counterparts). Nevertheless, we may consider the maximal topologically enriched groupoid, with its associated non-contractible space

$$\mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{\simeq}) \simeq \coprod \mathrm{BU}(n).$$

Endow $\mathrm{Vect}_{\mathbb{C}}$ with the symmetric monoidal direct sum, thereby making $\coprod \mathrm{BU}(n)$ into an $\mathbb{E}_\infty$-monoid, which in turn induces an $\mathbb{E}_\infty$-group structure on the group completion

$$(\coprod \mathrm{BU}(n))^{\mathrm{gpc}} \simeq \mathbb{Z}\times\mathrm{BU}.$$

Moving on to $\Omega^2(\mathbb{Z}\times\mathrm{BU})$, the $\mathbb{E}_2$-group structure comes from the double loop space structure. We see that the $\mathbb{E}_2$-structures are of, a priori, completely different nature.

In practice, it is more convenient to move the two structures to the same side. By the $\mathrm{B}\dashv\Omega$ adjunction, an $\mathbb{E}_2$-map

$$(\coprod \mathrm{BU}(n))^{\mathrm{gpc}} \simeq \mathbb{Z}\times\mathrm{BU} \longrightarrow \Omega^2(\mathbb{Z}\times\mathrm{BU})$$

is the same data as a (pointed) map

$$\mathrm{B}^2(\coprod \mathrm{BU}(n)) \longrightarrow \mathbb{Z}\times\mathrm{BU}.$$

Warning: this map is *not* an isomorphism – the left-hand side is 2-connective and the map is an isomorphism onto $\{0\}\times\mathrm{BU}$.

Our plan is to construct topological models for $\mathbb{Z}\times\mathrm{BU}$ and $\mathrm{B}^2(-)$, and the map.

1 Characterization of the Bott map

We begin with the formal characterization of the Bott map, up to multiplication by an integer. In this talk we will not determine the integer – this is left for future talks.

We would like to be able to study maps

$$\mathrm{B}^2(\coprod \mathrm{BU}(n)) \longrightarrow \mathbb{Z} \times \mathrm{BU}.$$

Note that $\mathrm{BU}(1)$ acts on everything in sight, corresponding to tensoring with a line bundle, and the map is equivariant with respect to this action. A non-coherent version of this equivariance already characterizes the Bott map.

Proposition 1. *Let $\theta\colon \mathrm{B}^2(\coprod \mathrm{BU}(n)) \longrightarrow \mathbb{Z} \times \mathrm{BU}$ be a map such that the following diagram*

$$\begin{tikzcd} \mathrm{BU}(1) \times \mathrm{B}^2(\coprod \mathrm{BU}(n)) \arrow[r, "\mathrm{Id}\times\theta"] \arrow[d] & \mathrm{BU}(1) \times \mathbb{Z} \times \mathrm{BU} \arrow[d] \\ \mathrm{B}^2(\coprod \mathrm{BU}(n)) \arrow[r, "\theta"] & \mathbb{Z} \times \mathrm{BU} \end{tikzcd}$$

commutes, then θ is homotopic to an integer times the Bott map $q\beta$.

Proof. The Bott map produces an isomorphism

$$\mathrm{B}^2(\coprod \mathrm{BU}(n)) \xrightarrow{\simeq} \mathrm{BU},$$

which identifies our map with

$$\theta\colon \mathrm{BU} \longrightarrow \mathbb{Z} \times \mathrm{BU}.$$

Letting $\iota\colon \mathrm{BU} \hookrightarrow \mathbb{Z} \times \mathrm{BU}$ denote the inclusion, we need to show that $\theta \simeq q\iota$. Note that $\mathbb{Z} \times \mathrm{BU}$ represents topological K-theory, namely we need to show their equality as classes in $\mathrm{KU}^0(\mathrm{BU})$.

Let $\mathrm{KU}_{\mathbb{Q}}$ denote the rationalization, and denote by θ_n, ι_n the images in $\mathrm{KU}^0_{\mathbb{Q}}(\mathrm{BU}(n))$. We shall prove the following:

Claim. For all $n \geq 1$, there is a rational number q_n such that $\theta_n = q_n \iota_n$.

This will suffice: The compatibility over n implies that $q_n = q$ is independent of n. Moreover, the fact that

$$\mathbb{Z}[[t]] \simeq \mathrm{KU}^0(\mathrm{BU}(1)) \to \mathrm{KU}^0_{\mathbb{Q}}(\mathrm{BU}(1))$$

is injective implies that q is an integer. As $\mathrm{KU}^0(\mathrm{BU}) \xrightarrow{\simeq} \lim \mathrm{KU}^0(\mathrm{BU}(n))$ we get that $\theta = q\iota$.

Let us prove the claim. Fix n. Recall that $\mathrm{KU}^0_{\mathbb{Q}}(\mathrm{BU}(n)) \simeq \mathbb{Q}[[x_1, \ldots, x_n]]$, where x_i is the i-th Adams operation ψ^i applied to $[E_n] - n$, where E_n is the tautological vector bundle. The class ι_n is identified x_1, and θ_n is some formal power series

$$f(x_1, \ldots, x_n) = \sum a_{e_1,\ldots,e_n} x_1^{e_1} \cdots x_n^{e_n} \qquad \in \qquad \mathbb{Q}[[x_1, \ldots, x_n]].$$

We now show the vanishing of almost all coefficients, using the condition from the proposition. We similarly have $\mathrm{KU}^0_{\mathbb{Q}}(\mathrm{BU}(1) \times \mathrm{BU}(n)) \simeq \mathbb{Q}[[t, x_1, \ldots, x_n]]$ where $t = [E_1] - 1$. The commutativity of the square, together with the fact that Adams operations on lines bunles is given by $\psi^i([L]) = [L]^i$, translates to

$$f(x_1, \ldots, x_n) \cdot [E_1] = f(x_1[E_1]^1, \ldots, x_n[E_1]^n).$$

Comparing coefficients, we conclude that

$$a_{e_1,\ldots,e_n}(1+t) = a_{e_1,\ldots,e_n}(1+t)^{1e_1+\cdots+ne_n}.$$

Thus, $a_{e_1,\ldots,e_n}$ must vanish unless $e_1 = 1$ and $e_i = 0$ for $i > 1$, and we conclude that $\gamma_n = f(x_1, \ldots, x_n) = a_{1,0,\ldots,0}x_1 = a_{1,0,\ldots,0}\iota_n$, as required. □

2 Topological model for $\mathbb{Z} \times \mathrm{BU}$

Recall that $\mathbb{Z} \times \mathrm{BU}$ is the group completion $\mathrm{N}(\mathrm{Vect}^{\simeq}_{\mathbb{C}})^{\mathrm{gpc}}$. We'd like to implement the group completion on the topological side. Intuitively then, we need some category whose objects are formal differences $W - V$. In the end we would like to to identify $W - V$ and $(W \oplus U) - (V \oplus U)$, which complicates things.

Definition 2. We define the topologically enriched category $\mathrm{Vect}^{\pm}_{\mathbb{C}}$ as follows.

- Objects: a pair (V, W) of finite dimensional complex vector spaces.
- Morphisms: a morphism from (V, W) to (V', W') is a pair of injective maps $V \hookrightarrow V', W \hookrightarrow W'$ and a subspace $U \leq V' \oplus W'$ such that the maps

  $$V \oplus U \longrightarrow V', \qquad W \oplus U \longrightarrow W'$$

  are isomorphisms.

Observe that there is a morphism from (V, W) to (V', W') if and only if $\dim V' - \dim V = \dim W' - \dim W = d \geq 0$. The topology is the subspace topology from

$$\hom(V, V') \times \hom(W, W') \times \mathrm{Gr}_d(V' \oplus W').$$

The map from (V, W) to $(V \oplus U, W \oplus U)$ is by choosing $U \simeq \Delta_U \leq (V \oplus U) \oplus (W \oplus U)$. Another consequence of the above is that there are only maps between objects of the same virtual dimension $\dim W - \dim V$.

Note that this is not a groupoid, hence $\mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{\pm})$ is an ∞-category and not an ∞-groupoid. We now wish to describe the space $\mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{\pm})^{\mathrm{gpd}}$ obtained by inverting all morphisms. Observe that there is an obvious map

$$\mathrm{Vect}_{\mathbb{C}} \longrightarrow \mathrm{Vect}_{\mathbb{C}}^{\pm}, \qquad W \mapsto (0, W).$$

Proposition 3. *There is the dashed isomorphism, fitting into a commutative diagram*

$$\begin{tikzcd} \coprod \mathrm{BU}(n) \arrow[r] \arrow[d, "\wr"] & \mathbb{Z} \times \mathrm{BU} \arrow[d, dashed, "\wr"] \\ \mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{\simeq}) \arrow[r] & \mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{\pm})^{\mathrm{gpd}} \end{tikzcd}$$

Proof. Recall that there are only maps between objects of the same virtual dimension. Let's denote by $\mathrm{Vect}_{\mathbb{C}}^{(d)} \subset \mathrm{Vect}_{\mathbb{C}}^{\pm}$ the subcategory on those (V, W) with virtual dimension $\dim W - \dim V = d$. Our goal is to show that $\mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{(d)})^{\mathrm{gpd}} \simeq \mathrm{BU}$.

We define an auxiliary topologically enriched category $\mathcal{C}$ as follows.

- Objects: vector spaces V of dimension $\dim V \geq d$.
- Morphisms: a morphism from V to W is a an injection $f \colon V \hookrightarrow W$ together with n elements $w_1, \ldots, w_d \in W$ whose image in $W/f(V)$ form a basis.

Claim. $\mathrm{N}(\mathcal{C})^{\mathrm{gpd}} \simeq \mathrm{BU}$.

Claim. $\mathrm{N}(\mathcal{C})^{\mathrm{gpd}} \simeq \mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{(d)})^{\mathrm{gpd}}$.

Let's start with the first claim. We have a map $\dim \colon \mathcal{C} \to \mathbb{Z}_{\geq d}$, inducing $\dim \colon \mathrm{N}(\mathcal{C}) \to \mathbb{Z}_{\geq d}$. The fiber over n is $\mathrm{BU}(n)$. Letting $F \colon \mathbb{Z}_{\geq d} \to \mathrm{Cat}_\infty$ denote the unstraightening, we see that

$$\mathrm{N}(\mathcal{C})^{\mathrm{gpd}} = \mathrm{colim}(F^{\mathrm{gpd}}) = \mathrm{colim}(\mathrm{BU}(n)) = \mathrm{BU}.$$

Moving on to the second claim, observe that we have a functor $\mathcal{C} \to \mathrm{Vect}_{\mathbb{C}}^{(d)}$ sending V to $(\mathbb{C}^{\dim V - d}, V)$, and we will show that $\mathrm{N}(\mathcal{C}) \to \mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{(d)})$ is cofinal, hence induces an isomorphism on groupoidifications. We use Quillen's theorem A, we need to show that for any $(V, W) \in \mathrm{Vect}_{\mathbb{C}}^{(d)}$, the slice

$$\mathcal{D} := \mathrm{N}(\mathcal{C}) \times_{\mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{(d)})} \mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{(d)})_{(V,W)/}$$

is contractible. Consider the composition $\dim \colon \mathcal{D} \to \mathrm{N}(\mathcal{C}) \to \mathbb{Z}_{\geq d}$. Arguing as above, $\mathcal{D}^{\mathrm{gpd}}$ is the colimit of the fibers, so it suffices to show that the become more and more connected. The fiber over n is some object of $\mathrm{N}(\mathcal{C})$, i.e. $\mathbb{C}^n$ up to isomorphism, together with a map, that is

$$\mathrm{Map}((V,W),(\mathbb{C}^{\dim V-n},\mathbb{C}^n))/GL_n(\mathbb{C}^n),$$

which one computes to see is $2(n-\dim V)$-connected (for $n \geq \dim V$), as required. $\square$

3 Topological model for double bar construction

Recall that in Avital's (second) talk, she introduced a way to model $\mathbb{E}_2$-algebras.

Definition 4. We define the category $\mathcal{O}^{\otimes}$ with:

- Objects: $(D_1,\dots,D_m)$ tuples of disks in $\mathbb{C}$.
- Morphisms: a morphism to $(D'_1,\dots,D'_n)$ is a pointed morphism

  $$\alpha\colon \{1,\dots,m,*\} \longrightarrow \{1,\dots,n,*\}$$

  such that for each $1 \leq j \leq n$ the disks $\{D_i\}_{i\in\alpha^{-1}(j)}$ are pairwise disjoint and contained in D'_j.

Proposition 5. *Let $\mathcal{E}$ be a symmetric monoidal ∞-category, then there is a fully faithful embedding*

$$\mathrm{Alg}_{\mathbb{E}_2}(\mathcal{E}) \hookrightarrow \mathrm{Alg}_{\mathcal{O}}(\mathcal{E})$$

whose essential image is those $\mathcal{O}$-algebras A such that for any inclusion of disks $D \subseteq D'$, the map $A(D) \to A(D')$ is an isomorphism.

In particular, an $\mathbb{E}_2$-∞-category (resp. space) is the same data as a functor $A\colon \mathcal{O}^{\otimes} \to \mathrm{Cat}_\infty$ (resp. to $\mathcal{S}$) satisfying:

1. *The map $A(D_1,\dots,D_m) \to A(D_1)\times\cdots\times A(D_m)$ is an isomorphism.*
2. *For $D \subseteq D'$, the map $A(D) \to A(D')$ is an isomorphism.*

Our next task is to use this to describe the double bar construciton B^2A of such A.

Definition 6. We let $\mathcal{O}^{\circ} \subset \mathcal{O}^{\otimes}$ be the (non-full non-wide) subcategory on:

- Objects: $(D_1,\dots,D_m)$ pairwise disjoint.
- Morphisms: α is in if and only if for every i such that $0 \in D_i$, we have $\alpha(i) \neq *$.

Proposition 7. *Let $A\colon \mathcal{O}^{\otimes} \to \mathcal{S}$ represent some $\mathbb{E}_2$-space, then*

$$\mathrm{B}^2A \simeq \operatorname{colim} A|_{\mathcal{O}^{\circ}}.$$

Proof sketch. We begin by constructing a map $F \to \mathrm{B}^2$. Recall that $\mathrm{B} \dashv \Omega$. For $X \in \mathcal{S}_*$, the $\mathbb{E}_2$-space $\Omega^2 X : \mathcal{O}^{\otimes} \to \mathcal{S}$ is given by $\Omega^2 X(D_1, \ldots, D_m) = \prod \mathrm{Map}_*(D_i^c, X)$. Recalling that in $\mathcal{O}^{\circ}$ the disks are disjoint, we construct a map

$$\Omega^2 X|_{\mathcal{O}^{\circ}}(D_1, \ldots, D_m) \to X, \qquad (f_1, \ldots, f_m) \mapsto \begin{cases} f_i(0) & \exists i \colon 0 \in D_i \\ x_0 & \text{otherwise} \end{cases}$$

which assembles into a map $F(\Omega^2 X) = \operatorname{colim} \Omega^2 X|_{\mathcal{O}^{\circ}} \to X$. Apply to $X = \mathrm{B}^2 A$ to get $F \to \mathrm{B}^2$.

Next, we reduce to free $\mathbb{E}_2$-algebras. B^2 commutes with all colimits since it is a left adjoint. From the model, F commutes with sifted colimits. Recall $\mathbb{E}_2$-spaces are generated under sifted colimits from $A = \mathrm{Free}(S)$, ranging over finite sets S, so we are reduced to showing that the map is an isomorphism for such A.

For that case, we recall that the free $\mathbb{E}_2$-space on an space X is explicitly constructed as a configuration space of 2-dimensional disks labeled by X. This can also be identified with $\Omega^2 \Sigma^2 X_+$. We therefore wish to identify $\mathrm{B}^2 \mathrm{Free}(S) \simeq \Sigma^2(S_+)$, which is a certain colimit, with the colimit over $\mathcal{O}^{\circ}$ of the configuration space description of, which is done by certain cofinality arguments. $\square$

One advantage of this construction is that it makes it easier to build maps out of the double bar construction, since mapping out of a colimit is doable.

The next observation is that sometimes we can make this colimit live in the topological world, via unstraightening. The point is that for $F \colon I \to \mathrm{Cat}_\infty$ with unstraightening $\mathcal{E} \to I$, we have $\mathcal{E}^{\mathrm{gpd}} \simeq \operatorname{colim}(F^{\mathrm{gpd}})$. Thus, we wish to describe the unstraightening in topological situations.

Definition 8. Let $\mathcal{C}$ be a symmetric monoidal topologically enriched category. We define a topologically enriched category $\mathcal{O}[\mathcal{C}]^{\otimes}$ with:

- Objects: $((X_1, D_1), \ldots, (X_m, D_m))$ a tuple of objects of $\mathcal{C}$ and disks.
- Maps: α as above, and for each $1 \leq j \leq n$ a map
$$\bigotimes_{i \in \alpha^{-1}(j)} X_i \longrightarrow X'_j.$$

Clearly $\mathcal{O}[\mathcal{C}]^{\otimes}$ lives over $\mathcal{O}^{\otimes} = \mathcal{O}[*]^{\otimes}$, and $\mathrm{N}(\mathcal{O}[\mathcal{C}]^{\otimes})$ is the unstraightening of the corresponding $\mathcal{O}$-algebra.

Corollary 9. *Let $\mathcal{C}$ be a symmetric monoidal topologically enriched category, then*

$$\mathrm{B}^2(\mathrm{N}(\mathcal{C})^{\mathrm{gpd}}) \simeq \mathrm{N}(\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} \mathcal{O}[\mathcal{C}]^{\otimes})^{\mathrm{gpd}}.$$

Example 10. We have

$$\mathrm{B}^2(\coprod \mathrm{BU}(n)) \simeq \mathrm{B}^2(\mathrm{N}(\mathrm{Vect}_{\mathbb{C}}^{\simeq})) \simeq \mathrm{N}(\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} \mathcal{O}[\mathrm{Vect}_{\mathbb{C}}^{\simeq}])^{\mathrm{gpd}}.$$

4 Topological model for the Bott map

Recall that our goal is to construct a map

$$\mathrm{B}^2(\coprod \mathrm{BU}(n)) \longrightarrow \mathbb{Z} \times \mathrm{BU}.$$

We have already constructed topological models for the source and the target as

$$\mathcal{O}^\circ \times_{\mathcal{O}^\otimes} \mathcal{O}[\mathrm{Vect}_{\mathbb{C}}^{\simeq}]^\otimes, \qquad \mathrm{Vect}_{\mathbb{C}}^{\pm}.$$

Our next goal is to construct a suitable map, in fact to the opposite of the target. Since these are simply topologically enriched categories, this is quite a tractable thing to do.

Objects

Recall that an object of the target is simply a pair (V, W), thought of as $W - V$. Also, recall that $\mathrm{B}^2(\coprod \mathrm{BU}(n)) \simeq \mathrm{BU}$ is the 2-connective cover of $\mathbb{Z} \times \mathrm{BU}$, so the map should land in virtual dimension $0 \in \mathbb{Z}$.

An object of the source is $((V_1, D_1), \ldots, (V_m, D_m))$ (where the disks are pairwise disjoint). We simply map

$$((V_1, D_1), \ldots, (V_m, D_m)) \quad \mapsto \quad (\bigoplus V_i, \bigoplus V_i).$$

Morphisms

Starting again with the target, a morphism from (V, W) to (V', W') is a pair of injective maps $V \hookrightarrow V', W \hookrightarrow W'$ together with a subspace $U \leq V' \oplus W'$ such that the maps

$$V \oplus U \longrightarrow V', \qquad W \oplus U \longrightarrow W'$$

are isomorphisms.

Now, a map to $((V_1', D_1'), \ldots, (V_n', D_n'))$ is a pointed morphism

$$\alpha \colon \{1, \ldots, m, *\} \longrightarrow \{1, \ldots, n, *\}$$

(such that $\{D_i\}_{i \in \alpha^{-1}(j)}$ are pairwise disjoint and contained in D_j, and if $0 \in D_i$ then $\alpha(i) \neq *$), together with isomorphisms

$$f_j \colon \bigoplus_{i \in \alpha^{-1}(j)} V_i \xrightarrow{\sim} V_j'.$$

Note that we may have $f(i) = *$, hence no map from V_i. Thus these combine into an inclusion

$$\iota \colon \bigoplus V_j' \xrightarrow{\bigoplus f_j^{-1}} \bigoplus_{f(i) \neq *} V_i \hookrightarrow \bigoplus V_i,$$

which also explains why we take op in the target.

We are thus left accounting for $\bigoplus_{f(i)=*} V_i$. I am not certain why Lurie doesn't take $U = \Delta_{\bigoplus_{f(i)=*} V_i}$, and take the map

$$(\bigoplus V_j', \bigoplus V_j') \to (\bigoplus V_i, \bigoplus V_i)$$

to be the one determined by $(\iota, \iota, \Delta_{\bigoplus_{f(i)=*} V_i})$. Presumably this ends up not giving the Bott map. Instead, Lurie takes a certain modification.

Modification

Lurie defines a modification of $\mathcal{O}[\mathrm{Vect}_{\mathbb{C}}^{\simeq}]^{\otimes}$, which is not a topologically enriched category, but rather a topological category, that is, the collection of objects is also made into a topological space. Note that above the disks didn't play a role in the construction of the functor (of course, they affect the source, since the colimit is over $\mathcal{O}^{\circ}$ which is all about the disks). The new category is going to add endomorphisms of the vector spaces, with eigenvalues constrained by the disks.

Definition 11. Let $D \subset \mathbb{C}$ be a disk, and let V be a vector space. A D-endomorphism of V is an endomorphism whose eigenvalues are in D, which we collect into $\mathrm{End}_D(V) \subset \mathrm{End}(V)$.

Definition 12. We define a topological category $\overline{\mathrm{Vect}}_{\mathbb{C}}^{\simeq,\otimes}$ as follows.

- Objects: $((V_1, D_1, \phi_1), \ldots, (V_m, D_m, \phi_m))$ where ϕ_i is a D_i-endomorphism of V_i.
- Morphisms: a pointed morphism

  $$\alpha \colon \{1, \ldots, m, *\} \longrightarrow \{1, \ldots, n, *\}$$

  with $\{D_i\}_{i \in \alpha^{-1}(j)}$ disjoint and contained in D_j', isomorphisms

  $$f_j \colon \bigoplus_{i \in \alpha^{-1}(j)} V_i \xrightarrow{\sim} V_j',$$

  such that the following diagram commutes

  $$\begin{tikzcd}
  \bigoplus_{i\in\alpha^{-1}(j)} V_i \arrow[r, "\sim"] \arrow[d, "\bigoplus \phi_i"'] & V_j' \arrow[d, "\phi_j'"] \\
  \bigoplus_{i\in\alpha^{-1}(j)} V_i \arrow[r, "\sim"] & V_j'
  \end{tikzcd}$$

The topology is the minimal making the forgetful (forgetting ϕ's) to $\mathcal{O}[\mathrm{Vect}_{\mathbb{C}}^{\simeq}]^{\otimes}$ continuous, that is, the objects are discrete in the V and D directions, having the $\prod \mathrm{End}_{D_i}(V_i)$ topology, and the morphisms are similarly topologized via the f_j's as before.

It is quite clear that the map $\overline{\mathrm{Vect}_{\mathbb{C}}^{\simeq}}^{,\otimes} \to \mathcal{O}[\mathrm{Vect}_{\mathbb{C}}^{\simeq}]^{\otimes}$ induces an isomorphism on groupoidifications: All that needs to be checked is that $\mathrm{End}_D(V)$ is always contractible, and can explicitly write a deformation retract onto $z_0 \mathrm{Id}_V$ for any arbitrary $z_0 \in D$. Therefore, we may as well work with this new category, constructing a map

$$\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} \overline{\mathrm{Vect}_{\mathbb{C}}^{\simeq}}^{,\otimes} \longrightarrow \mathrm{Vect}_{\mathbb{C}}^{\pm,\mathrm{op}}.$$

We make small modifications. On objects, we still take

$$((V_1, D_1, \phi_1), \ldots, (V_m, D_m, \phi_m)) \quad \mapsto \quad (\bigoplus V_i, \bigoplus V_i).$$

On morphisms, we still let

$$\iota\colon \bigoplus V_j' \xrightarrow{\bigoplus f_j^{-1}} \bigoplus_{f(i) \neq *} V_i \hookrightarrow \bigoplus V_i.$$

However, we take $U \leq \bigoplus_{f(i)=*} V_i \oplus \bigoplus_{f(i)=*} V_i$ to be the graph

$$\Gamma_{\bigoplus_{f(i)=*} \phi_i} = \{((v_i), (\phi_i(v_i))\}.$$

Corollary 13. *This construction gives rise to a map of topological categories*

$$\mathcal{O}^{\circ} \times_{\mathcal{O}^{\otimes}} \overline{\mathrm{Vect}_{\mathbb{C}}^{\simeq}}^{,\otimes} \longrightarrow \mathrm{Vect}_{\mathbb{C}}^{\pm,\mathrm{op}},$$

which on $\mathrm{N}(-)^{\mathrm{gpd}}$ *induces a map*

$$\mathrm{B}^2(\coprod \mathrm{BU}(n)) \longrightarrow \mathbb{Z} \times \mathrm{BU}$$

homotopic to an integer multiple of the Bott map.

Proof. It is clear that all constructions are compatible with tensoring with a 1-dimensional vector space, so we are in the situation of the proposition from the beginning. $\square$